arXiv:1207.1755v1 [math.PR] 7 Jul 2012

Super-Brownian motion in random environment as a limit point of critical branching random walks in random environment

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Abstract

We focus on the existence and its characterization of limit for a certain critical branching random walks in time-space random environment in 1 dimension which was introduced in [1]. Each particle performs simple random walk on $\mathbb{Z}$ and branching mechanism depends on the time-space site. The weak limit points of this measure valued processes are characterized as a solution of the non-trivial martingale problem and called super-Brownian motions in random environment in [14].

AMS 2000 Subject Classification: Primary 60J68, 60J80, 60K37

Key words: Superprocesses in random environment, Stochastic differential equation, Branching random walks in random environment.

We denote by $(\Omega, \mathcal{F}, P)$ a probability space. Let $\mathbb{N} = \{0, 1, 2, \cdots\}$, $\mathbb{N}^* = \{1, 2, 3, \cdots\}$, and $\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$.

1 Introduction

Super-Brownian motion(SBM) is a measure valued process which was introduced by Dawson and Watanabe independently[4, 19] and is obtained as the limit of critical (or asymptotically critical) branching Brownian motion (or branching random walks). There are many books for introduction of super-Brownian motion [5, 8] and dealing with several aspects of it [6, 7, 11, 16]. Super-Brownian motion has a lot of relations to the physics or bibliography.

An example of the construction is the followings, where we always treat Euclid space as the space, $\mathbb{R}^d$ in this paper.

We assume that at time 0, there are N particles in $\mathbb{Z}^d$ as the 0-th generation's particle. Each of N particles chooses independently of each others a

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nearest neighbor site uniformly and moves there at time 1 and after it, each particle independently of each others either dies or split into two particles with probability 1/2 (1st generation). The newly produced particles in n-th generation perform in the same manner, that is each of them chooses independently of each others a nearest neighbor site uniformly and moves there at time at time $n+1$ and after it, each particle independently of each others either dies or split into 2 particles with probability 1/2.

Let $X_t^{(N)}(\cdot)$ be the measure-valued Markov processes defined by

$$X_t^{(N)}(B) = \frac{\sharp\left\{\text{particles in } B\sqrt{N} \text{ at } \lfloor tN \rfloor\text{-th generation at time } tN\right\}}{N},$$

where $B \in \mathcal{B}(\mathbb{R})$ are Borel sets in $\mathbb{R}^d$ and $B\sqrt{N} = \{x = y\sqrt{N} \text{ for } y \in B\}$. Then, under some conditions, they converges as $N \to \infty$ to a measure-valued processes, *super-Brownian motion*. In particular, the limit $X_t(\phi)$ is characterized as the unique solution of the martingale problem:

$$\begin{cases} \text{For all } \phi \in \mathcal{D}(\Delta), \\ \qquad Z_t(\phi) := X_t(\phi) - X_0(\phi) - \int_0^t \frac{1}{2d} X_s\left(\Delta\phi\right) ds \\ \text{is an } \mathcal{F}_t^X\text{-continuous square-integrable martingale} \\ \qquad Z_0(\phi) = 0 \ \text{ and} \\ \qquad \langle Z(\phi)\rangle_t = \int_0^t X_s(\phi^2) ds, \end{cases}$$

where $\nu(\phi) = \int \phi d\nu$ for any measure ν.

In this paper, we consider super-Brownian motion in random environment, which are introduced in [14]. Mytnik showed the existence and uniqueness of the scaling limit $X_t(\cdot)$ for a certain critical branching diffusion in random environment with some conditions. It is characterized as the unique solution of the martingale problem:

$$\begin{cases} \text{For all } \phi \in \mathcal{D}(\Delta), \\ \qquad Z_t(\phi) := X_t(\phi) - X_0(\phi) - \int_0^t \frac{1}{2} X_s\left(\Delta\phi\right) ds \\ \text{is an } \mathcal{F}_t^X\text{-continuous square-integrable martingale} \\ \qquad Z_0(\phi) = 0 \ \text{ and} \\ \qquad \langle Z(\phi)\rangle_t = \int_0^t X_s(\phi^2) ds + \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} g(x,y)\phi(x)\phi(y) X_s(dx) X_s(dy) ds, \end{cases} \tag{1.1}$$

where $g(\cdot,\cdot)$ is bounded continuous function in a certain class. In this paper, we construct a super-Brownian motion in random environment as a limit point of scaled branching random walks in random environment, which is a solution of (1.1) for $g(x,y) = \delta_{x,y}$ case.

2 Branching random walks in random environment

Before giving the system of the branching random walks in random environment, we introduce Ulam-Harris tree $\mathcal{T}$ to labeling the particles. We set $T_k = (\mathbb{N}^*)^{k+1}$ for $k \geq 1$. Then, Ulam-Harris tree $\mathcal{T}$ is defined by $\mathcal{T} = \bigcup_{k\geq 0} T_k$.

We will give a name to each particle by using elements of $\mathcal{T}$.

i) When there are N particles at the 0-th generation, we label them as $1, 2, \cdots, N \in T_0$.

ii) If the n-th generation particle $\mathbb{x} = (x_0, \cdots, x_n) \in T_n$ gives birth to $k_{\mathbb{x}}$ particles, then we name them as $(x_0, \cdots, x_n, 1), \cdots, (x_0, \cdots, x_n, k_{\mathbb{x}}) \in T_{n+1}$.

Thus, every particle in the branching systems has its own name in $\mathcal{T}$. We define $|\mathbb{x}|$ by its generation, that is if $\mathbb{x}$ is an element of T_k, then $|\mathbb{x}| = k$. For convenience, we denote by $|\mathbb{x} \wedge \mathbb{y}|$ the generation of the common ancestor of $\mathbb{x}$ and $\mathbb{y}$. If $\mathbb{x}$ and $\mathbb{y}$ have no common ancestor, then we define $|\mathbb{x} \wedge \mathbb{y}| = -\infty$.

In this paper, we consider branching random walks in random environment for each N defined as follows:

i) There are N particles at site 0 at time 0.

ii) Each particle located at site $x \in \mathbb{Z}$ at time n moves to a uniformly chosen nearest neighbor site at time $n+1$ and it is replaced by 2 particles with probability $\dfrac{1}{2} + \dfrac{\xi(n,x)}{2N^{1/4}}$ or vanishes with probability $\dfrac{1}{2} - \dfrac{\xi(n,x)}{2N^{1/4}}$, where $\{\xi(n,x) : (n,x) \in \mathbb{N} \times \mathbb{Z}\}$ are i.i.d. random variables taking values in $\{-1, 1\}$ with uniformly.

We denote by $B_n^{(N)}$ and $B_{n,x}^{(N)}$ the total number of particles at time n and the local number of particles at site x at time n. In this paper, we focus on the scaled measure valued processes $X_t^{(N)}$ associated to this branching random walks:

$$X_0^{(N)} = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}, \quad \text{where } x_i = 0 \text{ for } 1 \leq i \leq N$$

and

$$X_t^{(N)} = \frac{1}{N}\sum_{i=1}^{B_{tN}^{(N)}} \delta_{x_i(t)/N^{1/2}}, \quad \text{for } t = \frac{1}{N}, \cdots, \frac{\lfloor KN \rfloor}{N} \text{ for each } K > 0,$$

where $x_i(t)$ is the position of the i-th particle at tN-th generation. We remark that if we identify $B_{tN,x}^{(N)}$ as the measure $B_{tN,x}^{(N)}\delta_x$, then $X_t^{(N)}$ is represented as

$$X_t^{(N)} = \frac{1}{N}\sum_{x\in\mathbb{Z}} B_{tN,x}^{(N)} \delta_{x/N^{1/2}} \quad \text{for } t = \frac{1}{N}, \cdots, \frac{\lfloor KN \rfloor}{N}.$$

Let $\mathcal{M}_F(\mathbb{R})$ be the set of the finite Borel measures on $\mathbb{R}$. For convenience, we extend this model to the cádlág paths in $\mathcal{M}_F(\mathbb{R})$ by

$$X_t^{(N)} = \frac{1}{N}\sum_{x\in\mathbb{Z}} B_{\underline{t}N,x}^{(N)}\delta_{x/N^{1/2}}, \quad \text{for } \underline{t} \le t < \underline{t} + \frac{1}{N},$$

where we define $\underline{t}$ for t and N by some positive number $\frac{i}{N}$ for some $i \in \mathbb{N}$ which satisfies $i/N \le t < (i+1)/N$. Then, $X_t^{(N)} \in \mathcal{M}_F(\mathbb{R})$ for each $t \in [0,K]$. Let $\phi \in \mathcal{B}_b(\mathbb{R})$, where $\mathcal{B}_b(\mathbb{R})$ is the set of the bounded Borel measurable functions on $\mathbb{R}$. We denote the product of $\nu \in \mathcal{M}_F(\mathbb{R})$ and $\phi \in \mathcal{B}_b(\mathbb{R})$ by $\nu(\phi)$, that is

$$\nu(\phi) = \int_{\mathbb{R}} \phi(x)\nu(dx).$$

Theorem 2.1. *We suppose $X_0^{(N)} = \frac{1}{N} \times N\delta_0$. Then, the sequence of measure valued processes $\left\{X_\cdot^{(N)} : N \in \mathbb{N}\right\}$ is tight and its weak limit point weakly is a continuous measure valued process $X_\cdot \in \mathcal{D}([0,\infty), \mathcal{M}_F(\mathbb{R}))$ which is a solution of the following martingale problem.*

$$\begin{cases} \textit{For all } \phi \in \mathcal{D}(\Delta), \\ \qquad Z_t(\phi) = X_t(\phi) - \phi(0) - \frac{1}{2}\int_0^t X_s(\Delta\phi)ds \\ \textit{is an } \mathcal{F}_t^X\textit{-continuous square-integrable martingale such that } Z_0(\phi) = 0 \textit{ and} \\ \qquad \langle Z(\phi)\rangle_t = \int_0^t X_s(\phi^2)ds \\ \qquad\qquad + \frac{1}{2}\int_0^t \int_{\mathbb{R}\times\mathbb{R}} \delta_{x-y}\phi(x)\phi(y)X_s(dx)X_s(dy)ds. \end{cases} \tag{2.1}$$

Remark: The martingale problem (2.1) is almost the same one mentioned beyond Theorem 3.1 in [14] for an exception. As mentioned there, we know that a limit process $X_t(dx)$ is almost surely absolutely continuous in t and x, that is there exists almost surely continuous random function $u(t,x)$ such that $X_t(dx) = u(t,x)dx$. Moreover, $u(t,x)$ is a weak solution of the following stochastic partial differential equation;

$$u_t = \frac{1}{2}u_{xx} + \sqrt{u + \frac{1}{2}u^2}\dot{W}, \tag{2.2}$$

where $\dot{W}$ is time-space white noise. SPDE (2.2) is considered in several papers on more general one. I particular, we know that pathwise uniqueness of (2.2) fails[12]. However, uniqueness in law remains unsolved.

Remark: The scaling factor of environment is $N^{-1/4}$ here which is different from the ones in [14], $N^{-1/2}$. Roughly speaking, the scaling factor in our model is determined by the followings. Now, the space is also scaled by $N^{-1/2}$. Then, the summation of the fluctuation of the first moment of offsprings in the segment $\{k\} \times [x,y]$ is $\displaystyle\sum_{z\in[xN^{1/2},yN^{1/2}]} \frac{\xi(k,z)}{N^{1/4}}$. Since it is the summation of i.i.d. random

variables of $(y-x)N^{1/2}$, the central limit theorem holds and it weakly converges to a Gaussian random variable with distribution $N(0,(y-x))$.

Now, we consider the value of $X_t^{(N)}(\phi)$. Since $X_t^{(N)}$ are constant in $t \in [\underline{t}, \underline{t}+\frac{1}{N})$, it is enough to see the difference between $X_{\underline{t}}^{(N)}$ and $X_{\underline{t}+\frac{1}{N}}^{(N)}$,

$$
\begin{aligned}
X_{\underline{t}+\frac{1}{N}}^{(N)}(\phi) - X_{\underline{t}}^{(N)}(\phi) &= \frac{1}{N}\sum_{\mathtt{x}\sim\underline{t}} \phi\left(\frac{Y_{\underline{t}N+1}^{\mathtt{x}}}{N^{1/2}}\right)\left(V^{\mathtt{x}} - 1 - \frac{\xi(\underline{t}N, Y_{\underline{t}N}^{\mathtt{x}})}{N^{1/4}}\right) \\
&+ \frac{1}{N}\sum_{\mathtt{x}\sim\underline{t}} \phi\left(\frac{Y_{\underline{t}N+1}^{\mathtt{x}}}{N^{1/2}}\right)\frac{\xi\left(\underline{t}N, Y_{\underline{t}N}^{\mathtt{x}}\right)}{N^{1/4}} \\
&+ \frac{1}{N}\sum_{\mathtt{x}\sim\underline{t}}\left(\phi\left(\frac{Y_{\underline{t}N+1}^{\mathtt{x}}}{N^{1/2}}\right) - \phi\left(\frac{Y_{\underline{t}N}^{\mathtt{x}}}{N^{1/2}}\right) - \frac{\phi\left(\frac{Y_{\underline{t}N}^{\mathtt{x}}+1}{N^{1/2}}\right) + \phi\left(\frac{Y_{\underline{t}N}^{\mathtt{x}}-1}{N^{1/2}}\right) - 2\phi\left(\frac{Y_{\underline{t}N}^{\mathtt{x}}}{N^{1/2}}\right)}{2}\right) \\
&+ \frac{1}{N}\sum_{\mathtt{x}\sim\underline{t}}\frac{\phi\left(\frac{Y_{\underline{t}N}^{\mathtt{x}}+1}{N^{1/2}}\right) + \phi\left(\frac{Y_{\underline{t}N}^{\mathtt{x}}-1}{N^{1/2}}\right) - 2\phi\left(\frac{Y_{\underline{t}N}^{\mathtt{x}}}{N^{1/2}}\right)}{2},
\end{aligned}
$$

where $\mathtt{x} \sim \underline{t}$ means that the particle $\mathtt{x}$ is the $\underline{t}N$-th generation, $Y_{\underline{t}N}^{\mathtt{x}}$ is the position of the particle $\mathtt{x}$ at time $\underline{t}N$ for $\mathtt{x} \sim \underline{t}N$ and $Y_{\underline{t}N+1}^{\mathtt{x}} = Y_{\underline{t}N+1}^{\mathtt{y}}$ for $\mathtt{y}$, a child of $\mathtt{x}$, $V^{\mathtt{x}}$ is the number of children of $\mathtt{x}$ and for simplicity, we omit N.

Thus, we have that for $0 \le \underline{t} \le t < \underline{t} + \frac{1}{N}$

$$
X_t^{(N)}(\phi) - X_0^{(N)}(\phi) = \left(M_t^{b,N}(\phi) + M_t^{e,N}(\phi) + M_t^{s,N}(\phi)\right) + \int_0^{\underline{t}} X_s^{(N)}\left(A^N\phi\right)ds, \tag{2.3}
$$

where

$$
\begin{aligned}
M_t^{(b,N)}(\phi) &= \frac{1}{N}\sum_{\underline{s}<t}\sum_{\mathtt{x}\sim\underline{s}} \phi\left(\frac{Y_{\underline{s}N+1}^{\mathtt{x}}}{N^{1/2}}\right)\left(V^{\mathtt{x}} - 1 - \frac{\xi(\underline{s}N, Y_{\underline{s}N}^{\mathtt{x}})}{N^{1/4}}\right), \\
M_t^{(e,N)}(\phi) &= \frac{1}{N}\sum_{\underline{s}<t}\sum_{\mathtt{x}\sim\underline{s}} \phi\left(\frac{Y_{\underline{s}N+1}^{\mathtt{x}}}{N^{1/2}}\right)\frac{\xi\left(\underline{s}N, Y_{\underline{s}N}^{\mathtt{x}}\right)}{N^{1/4}}, \\
M_t^{(s,N)}(\phi) &= \frac{1}{N}\sum_{\underline{s}<t}\sum_{\mathtt{x}\sim\underline{s}}\left(\phi\left(\frac{Y_{\underline{s}N+1}^{\mathtt{x}}}{N^{1/2}}\right) - \phi\left(\frac{Y_{\underline{s}N}^{\mathtt{x}}}{N^{1/2}}\right) - \frac{\phi\left(\frac{Y_{\underline{s}N}^{\mathtt{x}}+1}{N^{1/2}}\right) + \phi\left(\frac{Y_{\underline{s}N}^{\mathtt{x}}-1}{N^{1/2}}\right) - 2\phi\left(\frac{Y_{\underline{s}N}^{\mathtt{x}}}{N^{1/2}}\right)}{2}\right),
\end{aligned}
$$

and $A^N : \mathcal{B}_b(\mathbb{R}) \to \mathcal{B}_b(\mathbb{R})$ is the following operator;

$$
A^N\phi(x) = \frac{\phi\left(x + \frac{1}{N^{1/2}}\right) + \phi\left(x - \frac{1}{N^{1/2}}\right) - 2\phi(x)}{\frac{2}{N}}.
$$

Actually, we have that

$$\int_0^{\underline{t}} X_s^{(N)} \left(A^N \phi\right) ds = \sum_{\underline{s}<t} \sum_{\mathtt{x}\sim\underline{s}} \frac{1}{N} A^N \phi \left(\frac{Y^{\mathtt{x}}_{\underline{s}N}}{N^{1/2}}\right).$$

Also, we remark that $M_{\underline{t}}^{(b,N)}(\phi)$, $M_{\underline{t}}^{(e,N)}(\phi)$, and $M_{\underline{t}}^{(s,N)}(\phi)$ are $\mathcal{F}^{(N)}_{\underline{t}N}$-martingales, where $\mathcal{F}^{(N)}_n$ is the sigma algebra

$$\sigma\left(V^{\mathtt{x}}, Y^{\mathtt{x}}_{k+1}, \xi(k,x) : |\mathtt{x}| \leq n-1, k \leq n-1, x \in \mathbb{Z}\right),$$

where $\mathcal{F}^{(N)}_0 = \{\emptyset, \Omega\}$. Indeed, since $Y^{\mathtt{x}}_{n+1}$ are independent of $V^{\mathtt{x}}$ and $\xi(n,x)$,

$$\begin{aligned}
&E\left[M_{\underline{t}}^{(b,N)}(\phi) - M_{\underline{t}-1/N}^{(b,N)}(\phi) \middle| \mathcal{F}^{(N)}_{\underline{t}N-1}\right] \\
&\qquad = \frac{1}{N}\sum_{\mathtt{x}\sim\underline{t}-1/N} E\left[\phi\left(\frac{Y^{\mathtt{x}}_{\underline{t}N}}{N^{1/2}}\right)\middle| \mathcal{F}^{(N)}_{\underline{t}N-1}\right] E\left[V^{\mathtt{x}} - 1 - \frac{\xi\left(\underline{t}N-1, Y^{\mathtt{x}}_{\underline{t}N-1}\right)}{N^{1/4}} \middle| \mathcal{F}^{(N)}_{\underline{t}N-1}\right] = 0, \\
&E\left[M_{\underline{t}}^{(e,N)}(\phi) - M_{\underline{t}-1/N}^{(e,N)}(\phi) \middle| \mathcal{F}^{(N)}_{\underline{t}N-1}\right] \\
&\qquad = \frac{1}{N}\sum_{\mathtt{x}\sim\underline{t}-1/N} E\left[\phi\left(\frac{Y^{\mathtt{x}}_{\underline{t}N}}{N^{1/2}}\right)\middle| \mathcal{F}^{(N)}_{\underline{t}N-1}\right] E\left[\frac{\xi\left(\underline{t}N-1, Y^{\mathtt{x}}_{\underline{t}N-1}\right)}{N^{1/4}} \middle| \mathcal{F}^{(N)}_{\underline{t}N-1}\right] = 0,
\end{aligned}$$

and

$$E\left[M_{\underline{t}}^{(s,N)}(\phi) - M_{\underline{t}-1/N}^{(s,N)}(\phi) \middle| \mathcal{F}^{(N)}_{\underline{t}N-1}\right] = 0.$$

Moreover, we introduce another measure valued process, *historical process*:

$$H_t^{(N)} = \frac{1}{N}\sum_{\mathtt{x}\sim\underline{t}} \delta_{\frac{Y^{\mathtt{x}}_{\cdot\wedge t}}{N^{1/2}}}, \quad \in \mathcal{M}_F\left(D([0,\infty),\mathbb{R})\right).$$

where $Y^{\mathtt{x}}_s = Y^{\mathtt{y}}_s$ for $0 \leq s < |\mathtt{x}\wedge\mathtt{y}| + 1$. That is $Y^{\mathtt{x}}_s$ is the position of the $\lfloor sN \rfloor$-generation's ancestor of $\mathtt{x}$.

3 Proof of Theorem 2.1

In this section, we will give a proof of Theorem 2.1. The proof is divided into two steps:

i) Tightness.

ii) Identification of the limit point process.

3.1 Tightness

In this subsection, we will prove the following lemma.

Lemma 3.1. *The sequence $\{X^{(N)}\}$ is tight in $D([0,\infty),\mathcal{M}_F(\mathbb{R}))$, and each limit process is continuous.*

To prove it, we will use the following theorem which reduces the problem to the tightness of real-valued process. (See [16, Theorem II. 4. 1]).

Theorem 3.2. *Assume that E is a Polish space. Let D_0 be a separating class of $C_b(E)$ containing 1. A sequence of cádlág $\mathcal{M}_F(E)$-valued processes $\left\{X^{(N)}: N\in\mathbb{N}\right\}$ is C-relatively compact in $D\left([0,\infty),\mathcal{M}_F(E)\right)$ if and only if*

(i) for every $\varepsilon, T>0$, there is a compact set $K_{T,\varepsilon}$ in E such that

$$\sup_N P\left(\sup_{t\le T} X_t^{(N)}\left(K_{T,\varepsilon}^c\right)>\varepsilon\right)<\varepsilon,$$

(ii) and for all $\phi\in D_0$, $\left\{X^{(N)}(\phi): N\in\mathbb{N}\right\}$ is C-relatively compact in $D\left([0,\infty),\mathbb{R}\right)$.

Assumption: We choose $C_b^2(\mathbb{R})$ as D_0, where $C_b^2(\mathbb{R})$ is the set of bounded continuous function on $\mathbb{R}$ with bounded derivatives of order 1 and 2.

Hereafter, we prove the conditions (i) and (ii) of Theorem 3.2 for our case. In the beginning, we give the proof of (ii) by using the following lemmas:

Lemma 3.3. *For $\phi\in C_b^2(\mathbb{R})$, $\sup_{t\le K}\left|M_t^{(s,N)}(\phi)\right|\xrightarrow{L^2}0$ as $N\to\infty$ for all $K>0$.*

Lemma 3.4. *(See [16, Lemma II 4.5]) Let $\left(M_{\underline{t}}^{(N)},\overline{\mathcal{F}}_{\underline{t}}^N\right)$ be discrete time martingales with $M_0^{(N)}=0$.*
Let $\langle M^{(N)}\rangle_{\underline{t}}=\sum_{0\le\underline{s}<\underline{t}} E\left[\left(M_{\underline{s}+1/N}^{(N)}-M_{\underline{s}}^{(N)}\right)^2\middle|\overline{\mathcal{F}}_{\underline{s}}^N\right]$, and we extend $M_\cdot^{(N)}$ and $\langle M^{(N)}\rangle_\cdot$ to $[0,\infty)$ as right continuous step functions.

(i) If $\left\{\langle M^{(N)}\rangle_\cdot : N\in\mathbb{N}\right\}$ is C-relatively compact in $D([0,\infty),\mathbb{R})$ and

$$\sup_{0\le\underline{t}\le K}\left|M_{\underline{t}+1/N}^{(N)}-M_{\underline{t}}^{(N)}\right|\xrightarrow{P}0 \quad \text{as } N\to\infty \qquad \text{for all } K>0, \tag{3.1}$$

then $M_\cdot^{(N)}$ is C-relatively compact in $D([0,\infty),\mathbb{R})$.

If, in addition,

$$\left\{\left(M_{\underline{t}}^{(N)}\right)^2+\langle M^{(N)}\rangle_{\underline{t}}: N\in\mathbb{N}\right\} \qquad \text{is uniformly integrable for all } \underline{t}, \tag{3.2}$$

then $M^{(N_k)}\overset{w}{\Rightarrow}M$ in $D([0,\infty),\mathbb{R})$ implies M is a continuous L^2 martingale and $\left(M^{(N_k)},\langle M^{(N_k)}\rangle\right)\overset{w}{\Rightarrow}(M,\langle M\rangle)$ in $D([0,\infty),\mathbb{R})$.

Lemma 3.5. *For any $\phi \in C_b^2(\mathbb{R})$, the sequence $C_t^{(N)}(\phi) \equiv \int_0^{\underline{t}} X_s^{(N)}\left(A^N \phi\right) ds$ is C-relatively compact in $D([0,\infty),\mathbb{R})$.*

When we can verify the conditions of Lemma 3.4, the sequence $\left\{X^{(N)}(\phi) : N \in \mathbb{N}\right\}$ is C-relatively compact in $D([0,\infty),\mathbb{R})$. Moreover, if we check the condition of (i) in Theorem 3.2, then the tightness of $\left\{X^{(N)} : N \in \mathbb{N}\right\}$ follows immediately from Theorem 3.2.

So we start to prove the above lemmas now. First, we prepare the following lemma. It tells us the mean of the measure $X_{\underline{t}}^{(N)}$ is the same as the distribution of the scaled simple random walk.

Lemma 3.6. *If $\psi : D([0,\infty),\mathbb{R}) \to \mathbb{R}_{\geq 0}$ is Borel, then for any $t \geq 0$*

$$E\left[H_t^{(N)}(\psi)\right] = E_Y\left[\psi\left(\frac{Y_{(\cdot\wedge t)N}}{N^{1/2}}\right)\right], \tag{3.3}$$

where $Y.$ is the trajectory of simple random walk on $\mathbb{Z}$. In particular, for all $\phi \in \mathcal{B}_+(\mathbb{R})$,

$$E\left[X_{\underline{t}}^{(N)}(\phi)\right] = E_Y\left[\phi\left(\frac{Y_{\underline{t}N}}{N^{1/2}}\right)\right]. \tag{3.4}$$

Moreover, for all x, $K > 0$, we have that

$$P_Y\left(\sup_{t\leq K} X_t^{(N)}(1) \geq x\right) \leq x^{-1}. \tag{3.5}$$

Proof. (3.3) follows from the Markov property. Indeed, for $\underline{t} = 0$ the result holds and we assume that for $\underline{s} \leq \underline{t}$, the result holds. Then, we have

$$\begin{aligned}
E\left[H_{\underline{t}+1/N}^{(N)}(\psi)\right] &= E\left[\frac{1}{N}\sum_{\mathtt{y}\sim\underline{t}+1/N} \psi\left(\frac{Y^{\mathtt{y}}_{(\cdot\wedge(\underline{t}+1/N))N}}{N^{1/2}}\right)\right] \\
&= E\left[\frac{1}{N}\sum_{\mathtt{x}\sim\underline{t}} \psi\left(\frac{Y^{\mathtt{x}}_{(\cdot\wedge(\underline{t}+1/N))N}}{N^{1/2}}\right) E\left[V^{\mathtt{x}} \middle| \mathcal{F}_{\underline{t}N}^{(N)}\right]\right] \\
&= E\left[\frac{1}{N}\sum_{\mathtt{x}\sim\underline{t}} P_1\psi^{(N)}\left(Y^{\mathtt{x}}_{(\cdot\wedge\underline{t})N}\right)\right] \\
&= E_Y\left[\psi^{(N)}\left(Y_{\underline{t}N+1}\right)\right],
\end{aligned}$$

where $\psi^{(N)}(x) = \psi\left(\frac{x}{N^{1/2}}\right)$ and $P_1\psi(x) = E_Y^x[\psi(Y_1)]$. Also, (3.5) follows from the fact that $X_{\underline{t}}^{(N)}(1)$ is an $\mathcal{F}_{\underline{t}N}^{(N)}$-martingale and from the L^1 inequality for non-negative submartingales and (3.4). $\square$

Proof of Lemma 3.5. We know $X^{(N)}(\phi) = \phi(0)$. Also, we have that for any $K > 0$

$$\left| C_t^{(N)}(\phi) - C_s^{(N)}(\phi) \right| \le \int_{\underline{s}}^{\underline{t}} \left| X_{\underline{u}}^{(N)} \left(A^N \phi \right) \right| du$$
$$\le \sup_{\underline{u} \le K} C(\phi) X_{\underline{u}}^{(N)}(1) |\underline{t} - \underline{s}|. \tag{3.6}$$

We can use the Arzela-Ascoli Theorem by (3.5) and (3.6) so that $\left\{ C_\cdot^{(N)}(\phi) : N \in \mathbb{N} \right\}$ are C-relatively compact sequences in $D\left([0,\infty), \mathbb{R}\right)$. $\square$

Proof of Lemma 3.3. Let $h_N(y) = E^y \left[\left(\phi\left(\frac{Y_1}{N^{1/2}} \right) - \phi\left(\frac{Y_0}{N^{1/2}} \right) \right)^2 \right]$. By the orthogonality, we have that

$$E\left[\left(M_K^{(s,N)}(\phi) \right)^2 \right] = \frac{1}{N^2} \sum_{\underline{s} < K} E\left[\sum_{\mathbb{x} \sim \underline{s}} E\left[\left(\phi\left(\frac{Y^{\mathbb{x}}_{\underline{s}N+1}}{N^{1/2}} \right) - \phi\left(\frac{Y^{\mathbb{x}}_{\underline{s}N}}{N^{1/2}} \right) - \frac{\phi\left(\frac{Y^{\mathbb{x}}_{\underline{s}N}+1}{N^{1/2}} \right) + \phi\left(\frac{Y^{\mathbb{x}}_{\underline{s}N}-1}{N^{1/2}} \right) - 2\phi\left(\frac{Y^{\mathbb{x}}_{\underline{s}N}}{N^{1/2}} \right)}{2} \right)^2 \middle| \mathcal{F}^{(N)}_{\underline{s}N} \right] \right]$$
$$\le \frac{2}{N^2} \sum_{\underline{s} < K} E\left[\sum_{\mathbb{x} \sim \underline{s}} \left(h_N\left(\frac{Y^{\mathbb{x}}_{\underline{s}N}}{N^{1/2}} \right) + \frac{1}{N^2} \|A^N \phi\|^2 \right) \right]$$
$$\le 2E\left[\int_0^K \left(X_s^{(N)}(h_N) + \|A^N\phi\|^2 N^{-2} X_s^{(N)}(1) \right) ds \right]$$
$$\le 2\left(E_Y\left[\int_0^K \left(\phi\left(\frac{Y_{\underline{s}N+1}}{N^{1/2}} \right) - \phi\left(\frac{Y_{\underline{s}N}}{N^{1/2}} \right) \right)^2 ds \right] + \frac{K}{N^2} \sup_N \|A^N\phi\|^2 X_0^{(N)}(1) \right) \to 0.$$

We have used Lemma 3.6 and $\sup_N \|A^N \phi\| < \infty$ for $\phi \in C_b^2(\mathbb{R})$ in the last line. $\square$

Next, we will check the conditions in Lemma 3.4, that is,

(1) $\left\{ \left\langle M^{(b,N)}(\phi) \right\rangle_\cdot + \left\langle M^{(e,N)}(\phi) \right\rangle_\cdot : N \in \mathbb{N} \right\}$ is C-relatively compact in $D([0,\infty),\mathbb{R})$.

(2) $\sup_{0 \le \underline{t} \le K} \left| M^{(b,N)}_{\underline{t}+1/N}(\phi) - M^{(b,N)}_{\underline{t}}(\phi) + M^{(e,N)}_{\underline{t}+1/N}(\phi) - M^{(e,N)}_{\underline{t}}(\phi) \right| \xrightarrow{P} 0$ as $N \to \infty$ for all $K > 0$.

(3) $\left\{ \left(M^{(b,N)}_{\underline{t}}(\phi) \right)^2 + \left(M^{(e,N)}_{\underline{t}}(\phi) \right)^2 + \left\langle M^{(b,N)}(\phi) \right\rangle_{\underline{t}} + \left\langle M^{(e,N)}(\phi) \right\rangle_{\underline{t}} : N \in \mathbb{N} \right\}$ is uniformly integrable for all $\underline{t}$.

We remark that $M^{(b,N)}(\phi)$ and $M^{(e,N)}(\phi)$ are orthogonal and also under fixed environment $\{\xi(n,x) : (n,x) \in \mathbb{N} \times \mathbb{Z}\}$, $V^{\mathbb{x}}$ and $V^{\mathbb{y}}$ are independent for $\mathbb{x} \ne \mathbb{y}$. Therefore, we have that

$$\left\langle M^{(b,N)}(\phi) + M^{(e,N)}(\phi) \right\rangle_\cdot = \left\langle M^{(b,N)}(\phi) \right\rangle_\cdot + \left\langle M^{(e,N)}(\phi) \right\rangle_\cdot .$$

Indeed, we have that

$$
\begin{aligned}
&E\left[\left(M^{(b,N)}_{\underline{t}+1/N}(\phi)+M^{(e,N)}_{\underline{t}+1/N}(\phi)-M^{(b,N)}_{\underline{t}}(\phi)-M^{(e,N)}_{\underline{t}}(\phi)\right)^2\middle|\mathcal{F}^{(N)}_{\underline{t}N}\right]\\
&=\frac{1}{N^2}\sum_{\mathtt{x}\sim\underline{t}}E\left[\phi\left(\frac{Y^{\mathtt{x}}_{\underline{t}N+1}}{N^{1/2}}\right)^2\middle|\mathcal{F}^{(N)}_{\underline{t}N}\right]E\left[\left(V^{\mathtt{x}}-1-\frac{\xi\left(\underline{t}N,Y^{\mathtt{x}}_{\underline{t}N}\right)}{N^{1/4}}\right)^2\middle|\mathcal{F}^{(N)}_{\underline{t}N}\right]\\
&+\frac{1}{N^2}\sum_{\mathtt{x}\sim\underline{t},\tilde{\mathtt{x}}\sim\underline{t}}E\left[\phi\left(\frac{Y^{\mathtt{x}}_{\underline{t}N+1}}{N^{1/2}}\right)\phi\left(\frac{Y^{\tilde{\mathtt{x}}}_{\underline{t}N+1}}{N^{1/2}}\right)\middle|\mathcal{F}^{(N)}_{\underline{t}N}\right]\frac{\mathbf{1}\left\{Y^{\mathtt{x}}_{\underline{t}N}=Y^{\tilde{\mathtt{x}}}_{\underline{t}N}\right\}}{N^{1/2}}\\
&=\frac{1}{N}X^{(N)}_{\underline{t}}\left(\phi^2\right)\left(1-\frac{1}{N^{1/2}}\right)\left(1+\mathcal{O}(N^{-1/2})\right)\\
&+\frac{1}{N^2}\sum_{\mathtt{x},\tilde{\mathtt{x}}\sim\underline{t}}\phi\left(\frac{Y^{\mathtt{x}}_{\underline{t}N}}{N^{1/2}}\right)^2\frac{\mathbf{1}\left\{Y^{\mathtt{x}}_{\underline{t}N}=Y^{\tilde{\mathtt{x}}}_{\underline{t}N}\right\}}{N^{1/2}}\left(1+\mathcal{O}(N^{-1/2})\right),
\end{aligned}
$$

where $\left|\mathcal{O}(N^{-1/2})\right|\le C_\phi N^{-1/2}$ for a constant C_ϕ such that depends only on ϕ.

Thus,

$$
\begin{aligned}
\left\langle M^{(b,N)}(\phi)\right\rangle_{\underline{t}}+\left\langle M^{(e,N)}(\phi)\right\rangle_{\underline{t}}&=\left(\sum_{\underline{s}\le\underline{t}}\frac{1}{N}X^{(N)}_{\underline{s}}(\phi^2)\left(1-\frac{1}{N^{1/2}}\right)+\frac{1}{N}\sum_{\underline{s}\le\underline{t}}\sum_{x\in\mathbb{Z}}\phi\left(\frac{x}{N^{1/2}}\right)^2\frac{\left(B^{(N)}_{\underline{s}N,x}\right)^2}{N^{3/2}}\right)(1+\mathcal{O}(N^{-1/2}))\\
&=\left(1+\mathcal{O}(N^{-1/2})\right)\int_0^{\underline{t}}X^{(N)}_s(\phi^2)\left(1-\frac{1}{N^{1/2}}\right)ds\\
&\quad+\left(1+\mathcal{O}(N^{-1/2})\right)\int_0^{\underline{t}}\sum_{x\in\mathbb{Z}}\phi\left(\frac{x}{N^{1/2}}\right)^2\frac{\left(B^{(N)}_{\lfloor sN\rfloor,x}\right)^2}{N^{3/2}}ds.
\end{aligned}\tag{3.7}
$$

Therefore, we have that

$$
\begin{aligned}
&\left\langle M^{(b,N)}(\phi)\right\rangle_{\underline{t}}+\left\langle M^{(e,N)}(\phi)\right\rangle_{\underline{t}}-\left\langle M^{(b,N)}(\phi)\right\rangle_{\underline{s}}-\left\langle M^{(e,N)}(\phi)\right\rangle_{\underline{s}}\\
&\le\left(1+\mathcal{O}(N^{-1/2})\right)c^{(1)}_\phi\left(\sup_{\underline{u}\le K}X^{(N)}_{\underline{u}}(1)\right)|\underline{t}-\underline{s}|\\
&+\left(1+\mathcal{O}(N^{-1/2})\right)c^{(2)}_\phi\left(\sup_{u\le K}\sum_{x\in\mathbb{Z}}\frac{\left(B^{(N)}_{\lfloor sN\rfloor,x}\right)^2}{N^{3/2}}\right)|\underline{t}-\underline{s}|\\
&\le C\left(\left\langle M^{(b,N)}(1)\right\rangle_{\underline{t}}+\left\langle M^{(e,N)}(1)\right\rangle_{\underline{t}}-\left\langle M^{(b,N)}(1)\right\rangle_{\underline{s}}-\left\langle M^{(e,N)}(1)\right\rangle_{\underline{s}}\right)\\
&\le C\left(\sup_{\underline{u}\le K}\left\langle X^{(N)}(1)\right\rangle_{\underline{u}}\right)|\underline{t}-\underline{s}|.
\end{aligned}\tag{3.8}
$$

To prove the C-relatively compactness of (3.8), it is enough to show that

$$\sup_N \left(E\left[\left\langle X^{(N)}(1) \right\rangle_K \right] \right) = \sup_N E\left[\left(X^{(N)}_{\underline{K}}(1) \right)^2 \right] < \infty. \tag{3.9}$$

Lemma 3.7.

$$\sup_N E\left[\left(X^{(N)}_{\underline{K}}(1) \right)^2 \right] < \infty.$$

Proof. We remark that for each N, $B^{(N)}_n$ is a martingale with respect to the filtration $\mathcal{F}^{(N)}_n$.

Let $B^{(i,N)}_n$ be the total number of particles at time n which are the descendants from i-th initial particle. Then, we remark that for $i \neq j$

$$\begin{aligned} E\left[B^{(i,N)}_{\lfloor KN \rfloor} B^{(j,N)}_{\lfloor KN \rfloor} \right] &= E\left[E\left[B^{(i,N)}_{\lfloor KN \rfloor} \middle| \mathcal{F}^N_{\underline{K}N-1} \right] E\left[B^{(j,N)}_{\lfloor KN \rfloor} \middle| \mathcal{F}^N_{\underline{K}N-1} \right] \right] \\ &= E_{Y^1Y^2}\left[\left(1 + \frac{1}{N^{1/2}} \right)^{\sharp\left\{ i \leq \lfloor KN \rfloor : Y^1_i = Y^2_i \right\}} \right], \end{aligned} \tag{3.10}$$

where Y^1 and Y^2 are independent simple random walks on $\mathbb{Z}$ starting from the origin.

On the other hand,

$$\begin{aligned} E\left[\left(B^{(i,N)}_{\lfloor KN \rfloor} \right)^2 \right] &= 1 + \sum_{k=1}^{\lfloor KN \rfloor - 1} c E_{Y^1Y^2}\left[\left(1 + \frac{1}{N^2} \right)^{\sharp\left\{ k < i \leq \lfloor KN \rfloor : Y^1_i = Y^2_i \right\}} : Y^1_k = Y^2_k \right] + c \\ &\leq \lfloor KN \rfloor E_{Y^1Y^2}\left[\left(1 + \frac{1}{N^{1/2}} \right)^{\sharp\left\{ i \leq \lfloor KN \rfloor : Y^1_i = Y^2_i \right\}} \right], \end{aligned}$$

where $c = 1 - \frac{1}{N^{1/2}} < 1$ (See [20, Lemma 2.3]). Thus, we have that

$$\begin{aligned} E\left[\left(X^{(N)}_{\underline{K}}(1) \right)^2 \right] &\leq \frac{1}{N^2} \left(N(N-1) + N\lfloor KN \rfloor \right) E_{Y^1Y^2}\left[\left(1 + \frac{1}{N^{1/2}} \right)^{\sharp\left\{ i \leq \lfloor KN \rfloor : Y^1_i = Y^2_i \right\}} \right] \\ &\leq C(K) E_{Y^1Y^2}\left[\left(1 + \frac{1}{N^{1/2}} \right)^{\sharp\left\{ i \leq \lfloor KN \rfloor : Y^1_i = Y^2_i \right\}} \right]. \end{aligned} \tag{3.11}$$

Thus, if we prove $E_{Y^1Y^2}\left[\left(1 + \frac{1}{N^{1/2}} \right)^{\sharp\left\{ i \leq \lfloor KN \rfloor : Y^1_i = Y^2_i \right\}} \right]$ is bounded, then the proof is completed. The proof of its boundedness will be given later (Lemma 4.1). □

Also, we prove the following lemmas to check the conditions (1)-(3).

We denote by $\Delta M.$ martingale difference for martingale M, that is

$$\Delta M_{\underline{t}}^{(b,N)}(\phi) = \frac{1}{N}\sum_{\mathtt{x}\sim\underline{t}} \phi\left(\frac{Y^{\mathtt{x}}_{\underline{t}N+1}}{N^{1/2}}\right)\left(V^{\mathtt{x}} - 1 - \frac{\xi\left(\underline{t}N, Y^{\mathtt{x}}_{\underline{t}N}\right)}{N^{1/4}}\right)$$

$$\Delta M_{\underline{t}}^{(e,N)}(\phi) = \frac{1}{N}\sum_{\mathtt{x}\sim\underline{t}} \phi\left(\frac{Y^{\mathtt{x}}_{\underline{t}N+1}}{N^{1/2}}\right)\frac{\xi\left(\underline{t}N, Y^{\mathtt{x}}_{\underline{t}N}\right)}{N^{1/4}}. \tag{3.12}$$

Lemma 3.8. *For $\phi \in C_b^2(\mathbb{R})$,*

$$\lim_{N\to\infty} E\left[\sum_{\underline{t}\le K} |\Delta M_{\underline{t}}^{(b,N)}(\phi) + \Delta M_{\underline{t}}^{(e,N)}(\phi)|^4\right] = 0 \qquad \textit{for all } K>0. \tag{3.13}$$

Lemma 3.9. *For $\phi \in C_b^2(\mathbb{R})$,*

$$\sup_N E\left[\sup_{\underline{t}\le K}\left|M_{\underline{t}}^{(b,N)}(\phi) + M_{\underline{t}}^{(e,N)}(\phi)\right|^4\right] < \infty \qquad \textit{for all } K>0. \tag{3.14}$$

To prove Lemma 3.8, we will use the following proposition (See [2]).

Proposition 3.10. *Let $\phi : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is continuous, increasing, $\phi(0) = 0$ and $\phi(2\lambda) \le c_0\phi(\lambda)$ for all $\lambda \ge 0$. $(M_n, \mathcal{F}_n)$ is a martingale, $M_n^* = \sup_{k\le n}|M_k|$, $\langle M\rangle_n = \sum_{i=1}^n E\left[(M_k - M_{k-1})^2\middle|\mathcal{F}_{k-1}\right] + E[M_0^2]$, and $d_n^* = \max_{1\le k\le n}|M_k - M_{k-1}|$. Then, there exists $c = c(c_0)$ such that*

$$E\left[\phi\left(M_n^*\right)\right] \le cE\left[\phi\left(\langle M\rangle_n^{1/2}\right) + \phi\left(d_n^*\right)\right].$$

Proof of Lemma 3.8. It is enough to show that

$$\lim_{N\to\infty} E\left[\sum_{\underline{t}\le K}\left|\Delta M_{\underline{t}}^{(b,N)}(\phi)\right|^4 + \left|\Delta M_{\underline{t}}^{(e,N)}(\phi)\right|^4\right] = 0 \qquad \text{for all } K>0.$$

Conditional on $\mathcal{G}_{\underline{t}N}^{(N)} = \mathcal{F}_{\underline{t}N}^{(N)} \vee \sigma\left(\xi(n,x) : (n,x)\in\mathbb{N}\times\mathbb{Z}\right)$, $\Delta M_{\underline{t}}^{(b,N)}(\phi)$ is a sum of mean 0 independent random variables; $W^{(b,\mathtt{x},N)} := \frac{1}{N}\phi\left(\frac{Y^{\mathtt{x}}_{\underline{t}N+1}}{N^{1/2}}\right)\left(V^{\mathtt{x}} - 1 - \frac{\xi\left(\underline{t}N, Y^{\mathtt{x}}_{\underline{t}N}\right)}{N^{1/4}}\right)$.

Applying Proposition 3.10 into $\sum_{\mathtt{x}\sim\underline{t}} W^{(b,\mathtt{x},N)}$, we have

$$E\left[\left(\sup_{i\le B^{(N)}_{\underline{t}N}}\sum_{k=1}^{i} W^{(b,\mathtt{x}_k,N)}\right)^4\middle|\mathcal{G}^{(N)}_{\underline{t}N}\right] \le c\left(\sum_{\mathtt{x}\sim\underline{t}}\left(\frac{C_1(\phi)\left(1-\mathcal{O}(N^{-1/2})\right)}{N^2}\right)^2 + \left(\frac{C_2(\phi)}{N}\right)^4\right).$$

Thus,

$$E\left[\sum_{\underline{t}\le K}\left|\Delta M^{(b,N)}_{\underline{t}}(\phi)\right|^4\right]\le c\left(\frac{C_1(\phi)^2(1-\mathcal{O}(N^{-1/2}))}{N^4}\cdot(KN)\cdot E[NX^{(N)}_{\underline{t}}(1)]+KN\cdot\frac{C_2(\phi)^4}{N^4}\right)$$
$$\to 0.$$

Next, we will prove that

$$\lim_{N\to\infty}E\left[\sum_{\underline{t}\le K}\left|\Delta M^{(e,N)}_{\underline{t}}(\phi)\right|^4\right]=0\quad\text{for all } K>0.$$

It is clear that for $\phi\in C^2_b(\mathbb{R})$

$$E\left[\left|\Delta M^{(e,N)}_{\underline{t}}(\phi)\right|^4\right]\le C(\phi)E\left[\sum_{x,y\in\mathbb{Z}}2\frac{\left(B^{(N)}_{\underline{t}N,x}\right)^2\left(B^{(N)}_{\underline{t}N,y}\right)^2}{N^5}\right]$$

We will look at $B^{(N)}_{\underline{t}N,x}$ more closely as

$$B^{(N)}_{\underline{t}N,x}=\sum_{i=1}^{N}B^{(i,N)}_{\underline{t}N,x},$$

where $B^{(i,N)}_{\underline{t}N,x}$ is the numbers of particles located at x at time $\underline{t}N$ which are descendants from the i-th initial particle. Then,

$$\begin{aligned}E\left[\sum_{x,y\in\mathbb{Z}}\left(B^{(N)}_{\underline{t}N,x}\right)^2\left(B^{(N)}_{\underline{t}N,y}\right)^2\right]&=E\left[\sum_{x,y\in\mathbb{Z}}\left(\sum_{i=1}^{N}B^{(i,N)}_{\underline{t}N,x}\right)^2\left(\sum_{i=1}^{N}B^{(i,N)}_{\underline{t}N,y}\right)^2\right]\\&=\sum_{x,y\in\mathbb{Z}}E\left[\left(\sum_{i=1}^{N}\left(B^{(i,N)}_{\underline{t}N,x}\right)^2+\sum_{i\ne j}\left(B^{(i,N)}_{\underline{t}N,x}B^{(j,N)}_{\underline{t}N,x}\right)\right)\right.\\&\qquad\left.\left(\sum_{i=1}^{N}\left(B^{(i,N)}_{\underline{t}N,y}\right)^2+\sum_{i\ne j}\left(B^{(i,N)}_{\underline{t}N,y}B^{(j,N)}_{\underline{t}N,y}\right)\right)\right]\\&=\sum_{x,y\in\mathbb{Z}}E\left[\sum_{i^1,i^2,i^3,i^4=1}^{N}B^{(i^1,N)}_{\underline{t}N,x}B^{(i^2,N)}_{\underline{t}N,x}B^{(i^3,N)}_{\underline{t}N,y}B^{(i^4,N)}_{\underline{t}N,y}\right].\end{aligned}\tag{3.15}$$

There are several combination in the summation of (i^1,i^2,i^3,i^4):

(i) (4)-type, (ii) (3,1)-type (iii) (2,2)-type (iv) (2,1,1)-type (v) (1,1,1,1)-type.

These types represents the number of index which correspond in (i^1, i^2, i^3, i^4), for example $(i^1, i^2, i^3, i^4) = (3, 2, 4, 3)$ is $(2, 1, 1)$-type. When we look at the expectation in (3.15) for each type, we have that

$$
\begin{aligned}
&E\left[B^{(i^1,N)}_{\underline{t}N,x} B^{(i^2,N)}_{\underline{t}N,x} B^{(i^3,N)}_{\underline{t}N,y} B^{(i^4,N)}_{\underline{t}N,y}\right] \\
&\leq C\left((1\vee K)N\right)^{4-p} E_{Y^1Y^2Y^3Y^4}\left[E\left[\left(1+\frac{\xi(0,0)}{N^{1/4}}\right)^4\right]^{\sharp\{i\leq \underline{t}N : Y^a_i = Y^b_i, a\neq b\in\{1,\cdots,4\}\}} \mathbf{1}\left\{\begin{smallmatrix} Y^1_{\underline{t}N}=Y^2_{\underline{t}N}=x \\ Y^3_{\underline{t}N}=Y^4_{\underline{t}N}=y\end{smallmatrix}\right\}\right],
\end{aligned}
$$

where p is the number of blocks in each type, for example, (4)-type consists of one block, (2,2)-type consists of two blocks, and (2,1,1)-type consists of 3 blocks (See Corollary 4.3).

Also, it is clear that the number of combination for each type is order of N^p so that

$$
\begin{aligned}
(3.15) &\leq C(1\vee K)^4 N^4 \sum_{x,y\in\mathbb{Z}} E_{Y^1Y^2Y^3Y^4}\left[\left(1+\frac{10}{N^{1/2}}\right)^{\sharp\{i\leq \underline{t}N : Y^a_i = Y^b_i, a\neq b\in\{1,\cdots,4\}\}} \mathbf{1}\left\{\begin{smallmatrix} Y^1_{\underline{t}N}=Y^2_{\underline{t}N}=x \\ Y^3_{\underline{t}N}=Y^4_{\underline{t}N}=y\end{smallmatrix}\right\}\right] \\
&\leq C(1\vee K)^4 N^4 \sum_{x,y,z\in\mathbb{Z}} \prod_{\{a,b\}\in\{1,\cdots,4\}, a\neq b} E_{Y^1Y^2Y^3Y^4}\left[\left(1+\frac{11^6}{N^{1/2}}\right)^{\sharp\left\{i\leq \underline{t}N : Y^a_i = Y^b_i\right\}} \mathbf{1}\left\{\begin{smallmatrix} Y^1_{\underline{t}N}=Y^2_{\underline{t}N}=x \\ Y^3_{\underline{t}N}=Y^4_{\underline{t}N}=y\end{smallmatrix}\right\}\right]^{1/6} \\
&\leq C(1\vee K)^4 N^4 \sum_{x,y\in\mathbb{Z}} \prod_{\{a,b\}\in\{1,\cdots,4\}, a\neq b} \left(C(\underline{t})\prod_{i\neq a} P_{Y^i}\left(Y^i_{\underline{t}N} = x_i\right)\right)^{1/6} \\
&\leq C(1\vee K)^4 N^4 C(\underline{t}) \sum_{x,y\in\mathbb{Z}} (1\vee \underline{t}N)^{-1/2} P_{Y^1}\left(Y^1_{\underline{t}N} = x\right) P_{Y^2}\left(Y^2_{\underline{t}N} = y\right) \\
&\leq \frac{C(\underline{t})(1\vee K)^4 N^4}{(1\vee \underline{t}N)^{1/2}},
\end{aligned}
$$

where we have used Hölder's inequality and the fact that $(1+x)^k \leq 1 + 2^k x$ for $0\leq x\leq 1$ in the second inequality, and Lemma 4.1 in the third inequality.

Thus, we have that

$$
\begin{aligned}
\sum_{\underline{t}N\leq K} E\left[\left|\Delta M^{(e,N)}_{\underline{t}N}(\phi)\right|^4\right] &\leq \sum_{\underline{t}\leq K} C(K)C(\phi)\frac{(1\vee K)^4}{N(1\vee \underline{t}N)^{1/2}} \\
&\leq C(K)C(\phi)\frac{(KN)^{1/2}}{N} \to 0 \qquad \text{as } N\to\infty.
\end{aligned}
$$

□

Proof of Lemma 3.9. We apply Proposition 3.10 into martingale $M^{(b,N)}_{\underline{t}}(\phi) + M^{(e,N)}_{\underline{t}}(\phi)$. Then, we have that

$$
E\left[\sup_{\underline{t}\leq K}\left(M^{(b,N}_{\underline{t}}(\phi) + M^{(e,N)}_{\underline{t}}(\phi)\right)^4\right] \leq c\left(E\left[\left(\left\langle M^{(b,N)}(1)\right\rangle_K + \left\langle M^{(e,N)}(1)\right\rangle_K\right)^2\right]\right.
$$

$$+\sum_{\underline{t}\le K}\left(\left|\Delta M^{(b,N)}_{\underline{t}}(1)\right|^4+\left|\Delta M^{(e,N)}_{\underline{t}}(1)\right|^4\right)\Bigg).$$

The second term in the right hand side goes to 0 as $N\to\infty$ by Lemma 3.8. The first term is bounded above by

$$CE\left[\sum_{\underline{s},\underline{t}\le K}\left(\frac{X^{(N)}_{\underline{s}}(1)X^{(N)}_{\underline{t}}(1)}{N^2}+\sum_{x,y\in\mathbb{Z}}\frac{\left(B^{(N)}_{\underline{t}N,x}\right)^2}{N^{3/2}}\frac{\left(B^{(N)}_{\underline{t}'N,y}\right)^2}{N^{3/2}}\right)\right].$$

Since $X^{(N)}_{\underline{t}}(1)$ is a martingale, $E\left[X^{(N)}_{\underline{s}}(1)X^{(N)}_{\underline{t}}(1)\right]=E\left[X^{(N)}_{\underline{s}}(1)X^{(N)}_{\underline{s}}(1)\right]$ for $\underline{s}\le\underline{t}$. Thus,

$$E\left[\sum_{\underline{s},\underline{t}\le K}\frac{X^{(N)}_{\underline{s}}X^{(N)}_{\underline{t}}}{N^2}\right]\le K^2E\left[\left(X^{(N)}_{\underline{K}}(1)\right)^2\right]$$

is bounded in N for all K by Lemma 3.7.

Also, the same argument as the proof of Lemma 3.8 can be applied to

$$\sum_{\underline{s},\underline{t}\le K}E\left[\sum_{x,y\in\mathbb{Z}}\frac{\left(B^{(N)}_{\underline{s}N}\right)^2\left(B^{(N)}_{\underline{t}N,y}\right)^2}{N^5}\right]\le C\sum_{\underline{s},\underline{t}\le K}\frac{1}{N^5}\sum_{x,y\in\mathbb{Z}}E\left[\left(B^{(N)}_{\underline{s}N,x}\right)^4\right]^{1/2}E\left[\left(B^{(N)}_{\underline{t}N,x}\right)^4\right]^{1/2}$$

$$\le\frac{C}{N^5}\left(\sum_{\underline{s}\le K}\sum_{x}E\left[\left(B^{(N)}_{\underline{s}N,x}\right)^4\right]^{1/2}\right)^2$$

$$\le\frac{C(K\vee 1)^4}{N}\left(\sum_{\underline{t}\le K}\sum_{x\in\mathbb{Z}}E_{Y^1Y^2Y^3Y^4}\left[\left(1+\frac{10}{N^{1/2}}\right)^{\sharp\left\{u\le\underline{t}N:Y^a_u=Y^b_u,a\ne b\in\{1,2,3,4\}\right\}}\mathbf{1}\left\{\begin{smallmatrix}Y^1_{\underline{t}N}=Y^2_{\underline{t}N}=\\ Y^3_{\underline{t}N}=Y^4_{\underline{t}N}=x\end{smallmatrix}\right\}\right]^{1/2}\right)^2$$

$$\le\frac{C(K\vee 1)^4}{N}\left(\sum_{\underline{t}\le K}\sum_{x\in\mathbb{Z}}E_{Y^1Y^2}\left[\left(1+\frac{11^{12}}{N^{1/2}}\right)^{\sharp\{i\le\underline{t}N:Y^1_i=Y^2_i\}}\mathbf{1}\left\{Y^1_{\underline{t}N}=Y^2_{\underline{t}N}=x\right\}\right]^{1/2}P_{Y^1}\left(Y^1_{\underline{t}N}=x\right)\right)^2$$

$$\le\frac{C(K\vee 1)^4}{N}\left(\sum_{\underline{t}\le K}\sum_{x\in\mathbb{Z}}K^{1/4}N^{-1/4}P_{Y^1}\left(Y^1_{\underline{t}N}=x\right)^{3/2}\right)^2\quad(\because\text{ Lemma 4.1})$$

$$\le\frac{C(K)}{N^{3/2}}\left(\sum_{\underline{t}\le K}\frac{1}{(\underline{t}N\vee 1)^{1/4}}\right)^2<\infty.$$

□

Proof of the C-relatively compactness of $\{X^{(N)}_{\cdot}:N\in\mathbb{N}\}$. Ascoli-Arzela's theorem and (3.8) imply that $\left\{\left\langle M^{(b,N)}(\phi)+M^{(e,N)}(\phi)\right\rangle_{\cdot}:N\in\mathbb{N}\right\}$ is C-relatively

compact in $D([0,\infty),\mathbb{R})$. Also, (3.1) follows from Lemma 3.8 . The uniform integrability of $\left\{\left(M_{\underline{t}}^{(b,N)}+M_{\underline{t}}^{(e,N)}\right)^2+\left\langle M^{(b,N)}(\phi)+M^{(e,N)}(\phi)\right\rangle\right\}_{\underline{t}}$ has been shown by Lemma 3.7 and Lemma 3.9. Thus, we have checked all conditions in Lemma 3.4 so that $\left(M^{(b,N)}(\phi)+M^{(e,N)}(\phi),\left\langle M^{(b,N)}(\phi)+M^{(e,N)}(\phi)\right\rangle\right)$ is C-relatively compact in $D([0,\infty),\mathbb{R})$. Also, we have proved $C^{(N)}(\phi)$ is C-relatively compact in $D([0,\infty),\mathbb{R})$. Thus, $\left\{X_\cdot^{(N)}(\phi)\right\}$ is C-relatively compact in $D([0,\infty),\mathbb{R})$ for each $\phi\in C_b^2(\mathbb{R})$.

□

In the end of this subsection, we will check the condition (i) in Theorem 3.2. The proof follows the one in [16, p155]

Check for (i) in Theorem 3.2. Let $\varepsilon, T>0$ and $\eta(\varepsilon,T)>0$ (η will be chosen later). Let $K_0\subset D([0,\infty),\mathbb{R})$ be a compact set such that $\sup_N P\left(\frac{Y_{\cdot N}}{N^{1/2}}\in K_0^c\right)<\eta$. Let $K=\{y_t,y_{t-}:t\le T, y\in K_0\}$. Then, K is compact in $\mathbb{R}$. Clearly,

$$\sup_N P\left(\frac{Y_{Nt}}{N^{1/2}}\in K^c \text{ for some } t\le T\right)<\eta.$$

Let

$$\begin{aligned}R_t^{(N)}&=H_t^{(N)}\left(y:y(s)\in K^c \text{ for some } s\le t\right)\\&=\frac{1}{N}\sum_{\mathbb{x}\sim\underline{t}}\sup_{\underline{s}\le\underline{t}}\mathbf{1}_{K^c}\left(\frac{Y^{\mathbb{x}}_{\underline{s}N}}{N^{1/2}}\right).\end{aligned}$$

First, we will claim that $R_t^{(N)}$ is an $\mathcal{F}_{\underline{t}N}^{(N)}$-submartingale. Clearly, $R_\cdot^{(N)}$ is constant on $[\underline{t},\underline{t}+\frac{1}{N})$. So, it is enough to show that

$$E\left[R^{(N)}_{\underline{t}+\frac{1}{N}}-R^{(N)}_{\underline{t}}\,\middle|\,\mathcal{F}^{(N)}_{\underline{t}N}\right]\ge 0 \quad \text{a.s.} \tag{3.16}$$

We have

$$\begin{aligned}R^{(N)}_{\underline{t}+1/N}-R^{(N)}_{\underline{t}}&=\frac{1}{N}\sum_{\mathbb{x}\sim\underline{t}}\sup_{\underline{s}\le\underline{t}+1/N}\mathbf{1}_{K^c}\left(\frac{Y^{\mathbb{x}}_{\underline{s}N}}{N^{1/2}}\right)V^{\mathbb{x}}-\sup_{\underline{s}\le\underline{t}}\mathbf{1}_{K^c}\left(\frac{Y^{\mathbb{x}}_{\underline{s}N}}{N^{1/2}}\right)\\&\ge\frac{1}{N}\sum_{\mathbb{x}\sim\underline{t}}(V^{\mathbb{x}}-1)\sup_{\underline{s}\le\underline{t}}\mathbf{1}_{K^c}\left(\frac{Y^{\mathbb{x}}_{\underline{s}N}}{N^{1/2}}\right).\end{aligned}$$

The conditional expectation of the last term with respect to $\mathcal{F}^{(N)}_{\underline{t}N}$ is equal to 0. Thus, (3.16) is proved. Now we apply L^1-inequality for submartingale into $R_\cdot^{(N)}$ so that

$$P\left(\sup_{\underline{s}\le T}X^{(N)}_{\underline{s}}(K^c)>\varepsilon\right)\le P\left(\sup_{t\le T}R_t^{(N)}>\varepsilon\right)$$

$$\leq \varepsilon^{-1} E[R_{\underline{T}}^{(N)}]$$
$$\leq \varepsilon^{-1} P\left(\frac{Y_{sN}}{N^{1/2}} \in K^c, \text{ for some } s \leq T\right) \leq \varepsilon$$

by taking $\eta(T, \varepsilon) = \varepsilon^2$. $\square$

3.2 Identification of the limit point process

From the lemmas in section 3.1, we know that for $\phi \in C_b^2(\mathbb{R})$, each terms of

$$Z_{\underline{t}}^{(N)}(\phi) = X_{\underline{t}}^{(N)}(\phi) - \phi(0) - \int_0^{\underline{t}} X_{\underline{s}}^{(N)}(A^N \phi) ds, \tag{3.17}$$

and

$$\left\langle Z^{(N)}(\phi) \right\rangle_{\underline{t}} = \left\langle M^{(b,N)}(\phi) \right\rangle_{\underline{t}} + \left\langle M^{(e,N)}(\phi) \right\rangle_{\underline{t}} + \left\langle M^{(s,N)}(\phi) \right\rangle_{\underline{t}N} \tag{3.18}$$

are C-relatively compact in $D([0,\infty), \mathbb{R})$ and we found that the limit points satisfy

$$Z_t(\phi) = X_t(\phi) - \phi(0) - \int_0^t \frac{1}{2} X_s(\Delta \phi) ds$$

and

$$\langle Z(\phi) \rangle_t = \int_0^t X_s(\phi) ds + M_t^{(e)}(\phi),$$

where $M_t^{(e)}(\phi)$ is a limit point $M_{\underline{t}}^{(e,N)}(\phi)$. Thus, it enough to identify $M_t^{(e)}(\phi)$.

First, we give an approximation $X_t^{(N)}$ by some measure valued processes which have densities. For $(t,y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$, we define $u^{(N)}(t,y)$ by

$$u^{(N)}(t,y) = \frac{B_{tN,x}^{(N)}}{2\sqrt{N}} \qquad \text{for } \underline{t} \leq t < \underline{t} + \frac{1}{N} \text{ and } y \in \left[\frac{x-1}{N^{1/2}}, \frac{x+1}{N^{1/2}}\right), \ x \in \mathbb{Z}. \tag{3.19}$$

Actually, when we integrate (3.19) over $\mathbb{R}$ (or $[\frac{x-1}{N^{1/2}}, \frac{x+1}{N^{1/2}})$), they coincide with $X_t^{(N)}$ (or $\frac{B_{tN,x}^{(N)}}{N}$) and thus, we can regard $u^{(N)}(t,y)$ as an approximation of $X_\cdot^{(N)}$.

Then, $\langle M^{(e,N)}(\phi) \rangle_{\underline{t}}$ can be rewritten as

$$\begin{aligned}\left\langle M^{(e,N)}(\phi) \right\rangle_{\underline{t}} &= \int_0^{\underline{t}} \sum_{x \in \mathbb{Z}} \phi\left(\frac{x}{N^{1/2}}\right)^2 \frac{\left(B_{\lfloor sN \rfloor, x}^{(N)}\right)^2}{N^{3/2}} ds \\ &= \frac{(1 + \mathcal{O}(N^{-1/2}))}{2} \int_0^t \int_{y \in \mathbb{R}} \phi(y)^2 u^{(N)}(s,y)^2 dy ds.\end{aligned}$$

Therefore, we can conjecture that the limit point $M_t^{(e)}(\phi)$ is

$$\frac{1}{2}\int_0^t \int_{y\in\mathbb{R}} \phi^2(y)u(s,y)^2 dsdy \tag{3.20}$$

if $u^{(N)} \Rightarrow u$ for some $u(s,y)$ in some sense. In the following, we will check that (3.20) is true.

We denote by $\tilde{X}_t^{(N)}$ new measure-valued processes associated to $u^{(N)}(\cdot,\cdot)$, that is for $\phi \in C_b^2(\mathbb{R})$,

$$\tilde{X}_t^{(N)}(\phi) = \int_{\mathbb{R}} \phi(x)u^{(N)}(t,x)dx.$$

Then, it is clear that for $C_b^2(\mathbb{R})$ and for any $K > 0$

$$\limsup_{N\to\infty} E\left[\sup_{t<K}\left|\tilde{X}_t^{(N)}(\phi) - X_t^{(N)}(\phi)\right|\right] = 0.$$

Thus, $\left\{\tilde{X}_\cdot^{(N)} : N \in \mathbb{N}\right\}$ is C-relative compact in $D([0,\infty),\mathcal{M}_F(\mathbb{R}))$ and there are subsequences which weakly converges to $X_\cdot$, where $X_\cdot$ is the one given in (3.17).

In this section, we will prove the following lemma:

Lemma 3.11. *Let $X_\cdot$ be a limit point of the sequence $\{X_\cdot^{(N)} : N \in \mathbb{N}\}$. Then, the measure valued process $\{X_t(\cdot) : 0 \le t < \infty\}$ is almost surely absolutely continuous for all $t > 0$, that is there exists an adapted Borel-measurable-function-valued process $\{u_t : t > 0\}$ such that*

$$X_t(dx) = u_t(x)dx, \quad \textit{for all } t > 0,\ P\textit{-a.s.}$$

Define a sequences of measure valued processes $\left\{\mu_\cdot^{(N)}(dx) : N \in N\right\}$ by

$$\mu_t^{(N)}(dx) = \frac{1}{2}\int_0^t \left(u^{(N)}(s,x)\right)^2 dxds.$$

Lemma 3.12. *For any $\varepsilon > 0$ and for any $K > 0$, there exists a compact set $K^{\varepsilon,K} \subset \mathbb{R}$ such that*

$$\sup_N P\left(\sup_{t\le K} \mu_t^{(N)}\left(\left(K^{\varepsilon,K}\right)^c\right) > \varepsilon\right) < \varepsilon.$$

By using Lemma 3.11 and Lemma 3.12, we can complete the proof of Theorem 2.1 as follows:

Complete the proof of Theorem 2.1. We will verify that if $X_\cdot^{(N_k)}(dx) \Rightarrow u(\cdot,x)dx$ as $N_k \to \infty$, then

$$\mu_t^{(N)}(dx) \Rightarrow \frac{1}{2}\int_0^t u(s,x)^2 dxds. \tag{3.21}$$

Actually, we have already proved that $\left\{\left(\mu_t^{(N)}(\cdot)\right)_{t\in[0,\infty)} : N \in \mathbb{N}\right\}$ are C-relatively compact in $D\left([0,\infty), \mathcal{M}_F(\mathbb{R})\right)$ by checking the conditions in Theorem 3.2. However, we have already checked them in the proof of the tightness of $\{X_\cdot^{(N)} : N \in \mathbb{N}\}$ and Lemma 3.12. Thus, for any $\phi \in C_b^2(\mathbb{R})$,

$$\mu_t^{(N_k)}(\phi) \Rightarrow \mu_t(\phi).$$

Also, we may consider this convergence is almost surely, that is

$$\lim_{k\to\infty} \int_0^t \mu_s^{(N_k)}(\phi)\, ds = \int_0^t \mu_s(\phi) ds, \quad \text{a.s.} \tag{3.22}$$

Let $G_N(B,m)$ be the distributions of $u^{(N)}(t,x)$ for $B \in \mathcal{B}(\mathbb{R}_\geq \times \mathbb{R})$ and $m \in [0,\infty)$, that is

$$G_N(B,m) = \left|\left\{(t,x) \in B : u^{(N)}(t,x) \leq m\right\}\right|,$$

where $|\cdot|$ represents Lebesgue measure on $\mathbb{R}_\geq \times \mathbb{R}$. Then, the convergence of $u_t^{(N)}(\cdot)$ in (3.22) is equivalent to the convergence of the distributions $G_N(\cdot,\cdot)$.

Let $\mu_t^{(M,N)}(\cdot)$ be the truncated measure of $\mu_t^{(N)}(\cdot)$ for $M > 0$, that is

$$\mu_t^{(M,N)}(dx) = \frac{1}{2}\int_0^t \left(u^{(N)}(s,x) \wedge M\right)^2 dxds.$$

Then, it is clear that for any bounded function $C_{b,+}^2(\mathbb{R})$

$$\begin{aligned}\int_0^t \int_{\mathbb{R}} \phi(x)\left(u^{(N)}(s,x)\wedge M\right)^2 dxds = &\frac{1}{2}\int_0^t\int_{\mathbb{R}}\int_0^M \phi(x) m^2 G_N(dsdxdm)\\ &+\frac{1}{2}\int_0^t\int_{\mathbb{R}}\int_M^\infty \mathbf{1}_{\{u^{(N)}>M\}}(s,x)\phi(x)M^2 G_N(dsdxdm).\end{aligned}$$

The first term in the right hand side for N_k converges almost surely to

$$\int_0^t\int_{\mathbb{R}}\int_0^M \phi(x)m^2 G(dsdxdm) = \int_0^t\int_{\mathbb{R}} \phi(x)\left(u(s,x)\wedge M\right)^2 dxds,$$

where $G(\cdot,\cdot)$ is the distribution of $u(t,x)$. Thus, we have that for any $\phi \in C_{b,+}^2(\mathbb{R})$

$$\begin{aligned}\int_0^t\int_{\mathbb{R}}\phi(x)u(s,x)^2 dxds &= \lim_{M\to\infty}\lim_{N_k\to\infty}\int_0^t\int_{\mathbb{R}}\int_0^M \phi(x)m^2 G_N(dsdxdm)\\ &\leq \lim_{M\to\infty}\lim_{N_k\to\infty}\int_0^t\int_{\mathbb{R}}\phi(x)\left(u^{(N_k)}(t,x)\wedge M\right)^2 dx \leq \int_0^t \mu_s(\phi)ds, \quad \text{a.s.}\end{aligned}$$

Also, we know that for bounded function $\phi \in C_{b,+}^2(\mathbb{R})$, for any $t > 0$ and for any $\varepsilon > 0$

$$\lim_{M\to\infty}\sup_N P\left(\left|\int_0^t\int_{\mathbb{R}}\phi(x)\left(\left(u^{(N)}(s,x)\right)^2 - \left(u^{(N)}(s,x)\wedge M\right)^2\right)dxds\right| > \varepsilon\right)$$

$$\lim_{M\to\infty}\sup_N P\left(\left|\int_0^t\int_{\mathbb{R}}\mathbf{1}_{\{u^{(N)}>M\}}(s,x)\phi(x)m^2G_N(dsdxdm)\right|>\varepsilon\right)$$
$$=0,$$

by Lemma 3.9. Thus, for any bounded function $\phi\in C_{b,+}^2(\mathbb{R})$

$$\int_0^t\mu_s(\phi)ds=\lim_{N_k\to\infty}\int_0^t\int_{\mathbb{R}}\phi(x)\left(u^{(N_k)}(t,x)\right)^2dxds\le\int_0^t\int_{\mathbb{R}}\phi(x)u(t,x)^2dxds,\quad\text{in probability.}$$

This is true for $\phi\in C_b^2(\mathbb{R})$. Thus, we have proved (3.21). □

Proof of Lemma 3.12. First, we remark that $M_{\underline{t}}^{(e,N)}(\phi)$ is an $\mathcal{F}_{\underline{t}N}^{(N)}$-martingale even if $\phi(x)=\mathbf{1}_K(x)$ for Borel measurable set B. Then,

$$\left\langle M^{(e,N)}(K^c)\right\rangle_{\underline{t}}=\frac{1}{N}\sum_{\underline{s}<\underline{t}}\sum_{x\in K^cN^{1/2}}\frac{\left(B_{\underline{s}N,x}^{(N)}\right)^2}{N^{3/2}}=\frac{(1+\mathcal{O}(N^{-1/2}))}{2}\mu_t(K^c)$$

is a submartingale. Thus, we have that

$$\begin{aligned}P\left(\mu_t(K^c)>\varepsilon\right)&\le P\left(3\sup_{t\le T}\left\langle M^{(e,N)}(K^c)\right\rangle_{\underline{t}}>\varepsilon\right)\\&\le\varepsilon^{-1}E\left[\frac{3}{N}\sum_{\underline{s}<T}\sum_{x\in K^cN^{1/2}}\frac{\left(B_{\underline{s}N,x}^{(N)}\right)^2}{N^{3/2}}\right]\\&\le\varepsilon^{-1}C\sum_{\underline{s}<T}\sum_{x\in K^cN^{1/2}}(\underline{s}N)^{5/2}N^{-7/2}P_Y\left(Y_{\underline{s}N}=x\right)\\&\le\varepsilon^{-1}C\sup_{\underline{s}<T}P_Y\left(Y_{\underline{s}N}\in K^cN^{1/2}\right)\\&\le\varepsilon,\end{aligned}$$

by taking K^c as a compact set in $\mathbb{R}$ such that $C\sup_{\underline{s}<T}P_Y\left(Y_{\underline{s}N}\in K^cN^{1/2}\right)\le\varepsilon^2$, where we used Lemma 4.1 in the third inequality. □

In the rest of this section, we will prove Lemma 3.11.
For $\psi\in C_b^{1,2}([0,\infty)\times\mathbb{R},\mathbb{R})$, we define

$$X_t^{(N)}(\psi_t)=\sum_{\mathbb{x}\sim\underline{t}}\frac{\psi\left(t,\frac{Y_{\underline{t}N}^{\mathbb{x}}}{N^{1/2}}\right)}{N},\tag{3.23}$$

where $\psi_t(x)=\psi(t,x)$. Also, we have the following equation

$$X_{\underline{t}+\frac{1}{N}}^{(N)}\left(\psi_{\underline{t}+1/N}\right)-X_{\underline{t}}^{(N)}\left(\psi_{\underline{t}}\right)=\sum_{\mathbb{x}\sim\underline{t}}\frac{\psi\left(\underline{t}+1/N,\frac{Y_{\underline{t}N+1}^{\mathbb{x}}}{N^{1/2}}\right)}{N}\left(V^{\mathbb{x}}-1-\frac{\xi\left(\underline{t}N,Y_{\underline{t}N}^{\mathbb{x}}\right)}{N^{1/4}}\right)$$

$$
\begin{aligned}
&+\sum_{\mathtt{x}\sim\underline{t}} \frac{\psi\left(\underline{t}+1/N, \frac{Y^{\mathtt{x}}_{\underline{t}N+1}}{N^{1/2}}\right)}{N} \frac{\xi\left(\underline{t}N, Y^{\mathtt{x}}_{\underline{t}N}\right)}{N^{1/4}} \\
&+\sum_{\mathtt{x}\sim\underline{t}} \frac{2\psi\left(\underline{t}+1/N, \frac{Y^{\mathtt{x}}_{\underline{t}N+1}}{N^{1/2}}\right)-\psi\left(\underline{t}+1/N, \frac{Y^{\mathtt{x}}_{\underline{t}N}+1}{N^{1/2}}\right)-\psi\left(\underline{t}+1/N, \frac{Y^{\mathtt{x}}_{\underline{t}N}-1}{N^{1/2}}\right)}{2N} \\
&+\sum_{\mathtt{x}\sim\underline{t}} \frac{\psi\left(\underline{t}+\frac{1}{N}, \frac{Y^{\mathtt{x}}_{\underline{t}N}+1}{N^{1/2}}\right)+\psi\left(\underline{t}+\frac{1}{N}, \frac{Y^{\mathtt{x}}_{\underline{t}N}-1}{N^{1/2}}\right)-2\psi\left(\underline{t}, \frac{Y^{\mathtt{x}}_{\underline{t}N}}{N^{1/2}}\right)}{2N} \\
=:&\ \Delta M^{(b,N)}_{\underline{t}+1/N}(\psi_{\underline{t}+1/N})+\Delta M^{(e,N)}_{\underline{t}+1/N}(\psi_{\underline{t}+1/N}) \\
&+\Delta M^{(s,N)}_{\underline{t}+1/N}(\psi_{\underline{t}+1/N})+\Delta C^{(N)}_{\underline{t}+1/N}(\psi_{\underline{t}+1/N}).
\end{aligned}
$$

For $i=b,e,s$, $M^{(i,N)}_t(\psi_t)$ which are the sums of $\Delta M^{(i,N)}_t(\psi_t)$ up to t are martingales with respect to $\mathcal{F}^{(N)}_{\underline{t}N}$ as well as $M^{(i,N)}_{\cdot}(\phi)$ are.

We take ψ as the shift of

$$
\psi^x_t(y)=\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(y-x)^2}{2t}\right).
$$

Then, we have that for $\varepsilon,\varepsilon'>0$ and $t\geq\eta>0$

$$
\begin{aligned}
E\left[\left(X^{(N)}_t\left(\psi^x_\varepsilon\right)-X^{(N)}_t\left(\psi^x_{\varepsilon'}\right)\right)^2\right] \leq &\sum_{\underline{s}\leq\underline{t}} E\left[\left(\Delta M^{(b,N)}_{\underline{s}}\left(\psi^x_{t+\varepsilon-\underline{s}}-\psi^x_{t+\varepsilon'-\underline{s}}\right)\right)^2\right] && \text{(Mb)} \\
&+\sum_{\underline{s}\leq\underline{t}} E\left[\left(\Delta M^{(e,N)}_{\underline{s}}\left(\psi^x_{t+\varepsilon-\underline{s}}-\psi^x_{t+\varepsilon'-\underline{s}}\right)\right)^2\right] && \text{(Me)} \\
&+\sum_{\underline{s}\leq\underline{t}} E\left[\left(\Delta M^{(s,N)}_{\underline{s}}\left(\psi^x_{t+\varepsilon-\underline{s}}-\psi^x_{t+\varepsilon'-\underline{s}}\right)\right)^2\right] && \text{(Ms)} \\
&+E\left[\left(\sum_{\underline{s}\leq\underline{t}} \Delta C^{(N)}_{\underline{s}}\left(\psi^x_{t+\varepsilon-\underline{s}}-\psi^x_{t+\varepsilon'-\underline{s}}\right)\right)^2\right] && \text{(C)} \\
&+\left(\psi^x_{t+\varepsilon}(0)-\psi^x_{t+\varepsilon'}(0)\right)^2 && \text{(Initial term)} \\
&+E\left[\left(\sum_{\mathtt{x}\sim\underline{t}} \frac{\left(\psi^x_\varepsilon-\psi^x_{t+\varepsilon-\underline{t}}-\psi^x_{\varepsilon'}+\psi^x_{t+\varepsilon'-\underline{t}}\right)\left(\frac{Y^{\mathtt{x}}_{\underline{t}N}}{N^{1/2}}\right)}{2\sqrt{N}}\right)^2\right] && \text{(Error term)}
\end{aligned}
$$

Clearly, for fixed $\varepsilon > 0$, $\sup_y |\psi^x_\varepsilon(y) - \psi^x_{t+\varepsilon-\underline{t}}(y)| \leq \frac{C(\varepsilon)}{N}$. So (Error term) is bounded above by

$$E\left[\left(X^{(N)}\left(\frac{C(\varepsilon)+C(\varepsilon')}{N}\right)\right)^2\right] \to 0, \quad \text{as } N \to \infty.$$

Also,

$$(\text{Initial term}) \leq (\varepsilon - \varepsilon')^2 \left((t+\varepsilon) \wedge (t+\varepsilon')\right)^{-3},$$

where we have used [16, Lemma III 4.5 (a)], that is for $0 \leq \delta \leq p$,

$$|\psi^x_{t+\varepsilon}(y) - \psi^x_t(y)|^p \leq \left(\varepsilon t^{-3/2}\right)^\delta \left(\left(\psi^x_{t+\varepsilon}(y)\right)^{p-\delta} + \left(\psi^x_t(y)\right)^{p-\delta}\right) \tag{3.24}$$

for all $x, y \in \mathbb{R}$, $t > 0$, and $\varepsilon > 0$.

Lemma 3.13. *For $\varepsilon, \varepsilon' > 0$ and $t \geq \eta > 0$,*

$$\lim_{N\to\infty} E\left[\left(\sum_{\underline{s}\leq t} \Delta C^{(N)}_{\underline{s}}\left(\psi^x_{t+\varepsilon-\underline{s}} - \psi^x_{t+\varepsilon'-\underline{s}}\right)\right)^2\right] = 0.$$

Proof.

$$\begin{aligned}
\Delta C^{(N)}_{\underline{s}}\left(\psi^x_{t+\varepsilon-\underline{s}}\right) &= \sum_{\mathtt{x}\sim\underline{s}} \frac{\psi^x_{t+\varepsilon-\underline{s}-1/N}\left(\frac{Y^{\mathtt{x}}_{\underline{t}N}+1}{N^{1/2}}\right) + \psi^x_{t+\varepsilon-\underline{s}-1/N}\left(\frac{Y^{\mathtt{x}}_{\underline{t}N}-1}{N^{1/2}}\right) - \psi^x_{t+\varepsilon-\underline{s}}\left(\frac{Y^{\mathtt{x}}_{\underline{t}N+1}}{N^{1/2}}\right) - \psi^x_{t+\varepsilon-\underline{s}}\left(\frac{Y^{\mathtt{x}}_{\underline{t}N-1}}{N^{1/2}}\right)}{2N} \\
&\quad + \sum_{\mathtt{x}\sim\underline{s}} \frac{\psi^x_{t+\varepsilon-\underline{s}}\left(\frac{Y^{\mathtt{x}}_{\underline{t}N}+1}{N^{1/2}}\right) + \psi^x_{t+\varepsilon-\underline{s}}\left(\frac{Y^{\mathtt{x}}_{\underline{t}N}-1}{N^{1/2}}\right) - 2\psi^x_{t+\varepsilon-\underline{s}}\left(\frac{Y^{\mathtt{x}}_{\underline{t}N}}{N^{1/2}}\right)}{2N} \\
&\leq \sum_{\mathtt{x}\sim\underline{s}} \frac{1}{N^2}\left(\left.\frac{\partial\psi^x\left(t+\varepsilon-s, \frac{Y^{\mathtt{x}}_{\underline{s}N}}{N^{1/2}}\right)}{\partial s}\right|_{s=\underline{s}} + \mathcal{O}(N^{-1/2})\right) \\
&\quad + \sum_{\mathtt{x}\sim\underline{s}} \frac{1}{N^2}\left(\left.\frac{\partial^2\psi^x(t+\varepsilon-\underline{s}, y)}{2\partial y^2}\right|_{y=\frac{Y^{\mathtt{x}}_{\underline{s}N}}{N^{1/2}}} + \mathcal{O}(N^{-1/2})\right)
\end{aligned}$$

Since $\frac{\partial\psi^x(t+\varepsilon-s,y)}{\partial s} + \frac{\partial^2\psi^x(t+\varepsilon-s,y)}{2\partial y2} = 0$, we have that

$$\left|\Delta C^{(N)}_{\underline{s}}\left(\psi^x_{t+\varepsilon-\underline{s}}\right)\right| \leq C(\varepsilon,\eta)\frac{X^{(N)}_{\underline{s}}(1)}{N^{3/2}}.$$

Thus,

$$E\left[\left(\sum_{\underline{s}\leq t} \Delta C^{(N)}_{\underline{s}}\left(\psi^x_{t+\varepsilon-\underline{s}} - \psi^x_{t+\varepsilon'-\underline{s}}\right)\right)^2\right] \leq E\left[\left(C(\varepsilon,\eta)+C(\varepsilon',\eta)\right)^2 \sup_{\underline{s}\leq t}\left(\frac{X^{(N)}_{\underline{s}}(1)}{N^{1/2}}\right)^2\right]$$

$$\to 0 \ \text{ as } N \to \infty.$$

Indeed, for each N, $X^{(N)}_{\underline{s}}(1)$ is a martingale so that by L^2-maximum inequality and by Lemma 3.7,

$$\sup_N E\left[\sup_{\underline{s}\le \underline{t}} \left(X^{(N)}_{\underline{s}}(1)\right)^2\right] \le 4 \sup_N E\left[\left\langle X^{(N)}(1)\right\rangle_{\underline{t}}\right] < \infty.$$

□

Thus, we have by Fatou's lemma that

$$E\left[\left(X_t(\psi^x_\varepsilon) - X_t(\psi^x_{\varepsilon'})\right)^2\right] \le \liminf_{N_k\to\infty} \left((Mb) + (Me) + (Ms)\right).$$

Hereafter, we will see the right hand side .

Lemma 3.14. *Suppose $\varepsilon > \varepsilon' > 0$, $t \ge \eta > 0$, and $0 < \delta < \frac{1}{2}$. Then, for any $x \in \mathbb{R}$*

$$\limsup_{N\to\infty} (Mb) \le C_\delta (\varepsilon - \varepsilon')^\delta (t + \varepsilon')^{-\delta}.$$

Proof. By direct calculation, we have that for $\varepsilon > \varepsilon > 0'$, for $t \ge \eta > 0$, and for $0 < \delta < \frac{1}{2}$

(Mb)

$$= \left(1 - \frac{1}{N^{1/2}}\right) E\left[\sum_{\underline{s}\le\underline{t}} \sum_{z\in\mathbb{Z}} \frac{\left(\psi^x_{t+\varepsilon-\underline{s}}\left(\frac{z}{N^{1/2}}\right) - \psi^x_{t+\varepsilon'-\underline{s}}\left(\frac{z}{N^{1/2}}\right)\right)^2}{N^2} B^{(N)}_{\underline{s}N,z}\right]$$

$$\le E\left[\sum_{\underline{s}\le\underline{t}} \frac{\left(\psi^x_{t+\varepsilon-\underline{s}}\left(\frac{Y_{\underline{s}N}}{N^{1/2}}\right) - \psi^x_{t+\varepsilon'-\underline{s}}\left(\frac{Y_{\underline{s}N}}{N^{1/2}}\right)\right)^2}{N}\right] \qquad (\because \text{Lemma 3.6 })$$

$$\le \int_0^{\underline{t}} E\left[\left((\varepsilon - \varepsilon')(t + \varepsilon' - s)^{-3/2}\right)^\delta \left(\left(\psi^x_{t+\varepsilon-s}\left(\frac{Y_{\underline{s}N}}{N^{1/2}}\right)\right)^{2-\delta} + \left(\psi^x_{t+\varepsilon'-s}\left(\frac{Y_{\underline{s}N}}{N^{1/2}}\right)\right)^{2-\delta}\right)\right] ds \quad (\because (3.24)).$$

Thus,

$$\liminf_{N_k\to\infty} (\text{Mb})$$

$$\le \int_0^{\underline{t}} \int_{\mathbb{R}} \left((\varepsilon - \varepsilon')(t + \varepsilon' - s)^{-3/2}\right)^\delta \left(\left(\psi^x_{t+\varepsilon-s}(y)\right)^{2-\delta} + \left(\psi^x_{t+\varepsilon'-s}(y)\right)^{2-\delta}\right) \psi^0_s(y) dy ds \quad (\because \text{invariance principle})$$

$$\le (\varepsilon - \varepsilon')^\delta \int_0^t (t + \varepsilon' - s)^{-3\delta/2} (2-\delta)^{-1/2} \left((t + \varepsilon - s)^{\frac{\delta-1}{2}} \left(\frac{2-\delta}{t + \varepsilon + (1-\delta)s}\right)^{1/2}\right) ds$$

$$+ (\varepsilon - \varepsilon')^\delta \int_0^t (t + \varepsilon' - s)^{-3\delta/2} (2-\delta)^{-1/2} \left((t + \varepsilon' - s)^{\frac{\delta-1}{2}} \left(\frac{2-\delta}{t + \varepsilon' + (1-\delta)s}\right)^{1/2}\right) ds$$

$$\left(\because \int_{\mathbb{R}} \psi_s^x(y)\psi_t^0(y)dy = \psi_{t+s}^0(x)\right)$$

$$\leq C_\delta(\varepsilon-\varepsilon')^\delta (t+\varepsilon')^{-1/2}\int_0^t (t+\varepsilon'-s)^{-1/2-\delta}ds \leq C_\delta(\varepsilon-\varepsilon')^\delta (t+\varepsilon')^{-1/2}(t+\varepsilon')^{1/2-\delta}.$$

□

Lemma 3.15. *For all $x \in \mathbb{R}$, $\varepsilon > \varepsilon' > 0$, and $t \geq \eta > 0$, we have*

$$\lim_{N\to\infty} (Ms) = 0.$$

Proof. The proof is the same as the proof of Lemma 3.5. □

Lemma 3.16. *Suppose $\varepsilon > \varepsilon' > 0$, $t \geq \eta > 0$, and $0 < \delta < \frac{1}{2}$. Then, for any $x \in \mathbb{R}$*

$$\limsup_{N\to\infty} (Me) \leq C(t,\delta)t^{3/2}(\varepsilon-\varepsilon')^\delta \left(t+\varepsilon'\right)^{-\delta}.$$

Proof.

$$\begin{aligned}
&(\mathrm{Me}) \\
&\leq E\left[\sum_{\underline{s}\leq\underline{t}}\sum_{z\in\mathbb{Z}} \frac{\left(\psi^x_{t+\varepsilon-\underline{s}}\left(\frac{z}{N^{1/2}}\right)-\psi^x_{t+\varepsilon'-\underline{s}}\left(\frac{z}{N^{1/2}}\right)\right)^2}{N}\frac{\left(B^{(N)}_{\underline{s}N,z}\right)^2}{N^{3/2}}\right] \\
&\leq E_{Y,Y'}\left[\int_0^t \left(\psi^x_{t+\varepsilon-\underline{s}}\left(\frac{Y_{\underline{s}N}}{N^{1/2}}\right)-\psi^x_{t+\varepsilon-\underline{s}}\left(\frac{Y_{\underline{s}N}}{N^{1/2}}\right)\right)^2(\underline{s}\vee 1)\left(1+\frac{1}{N^{1/2}}\right)^{\sharp\left\{\underline{u}\leq\underline{s}:Y_{\underline{u}N}=Y'_{\underline{u}N}\right\}} N^{1/2}\mathbf{1}\left\{Y_{\underline{s}N}=Y'_{\underline{s}N}\right\}ds\right] \\
&\leq \sum_{z\in\mathbb{Z}} E_{Y,Y'}\left[\int_0^t \left(\psi^x_{t+\varepsilon-\underline{s}}\left(\frac{z}{N^{1/2}}\right)-\psi^x_{t+\varepsilon'-\underline{s}}\left(\frac{z}{N^{1/2}}\right)\right)^2(\underline{s}\vee 1)\left(1+\frac{1}{N^{1/2}}\right)^{\sharp\left\{\underline{u}\leq\underline{s}:Y_{\underline{u}N}=Y'_{\underline{u}N}\right\}} N^{1/2}\mathbf{1}\left\{Y_{\underline{s}N}=Y'_{\underline{s}N}=z\right\}ds\right] \\
&\leq C(t)t^{1/2}\int_0^t\sum_{z\in\mathbb{Z}}\left(\psi^x_{t+\varepsilon-\underline{s}}\left(\frac{z}{N^{1/2}}\right)-\psi^x_{t+\varepsilon'-\underline{s}}\left(\frac{z}{N^{1/2}}\right)\right)^2(s\vee 1)P\left(Y_{\underline{s}N}=z\right)ds \qquad (\because \text{Lemma 4.1}) \\
&\leq C(t)t^{1/2}\int_0^t\int_{\mathbb{R}}\left(\psi^x_{t+\varepsilon-\underline{s}}(y)-\psi^x_{t+\varepsilon'-\underline{s}}(y)\right)^2(s\vee 1)\psi^0_s(y)dyds \\
&\leq C(t)t^{1/2}\int_0^t\int_{\mathbb{R}}\left((\varepsilon-\varepsilon')(t+\varepsilon-s)^{-3/2}\right)^\delta\left(\left(\psi^x_{t+\varepsilon-s}(y)\right)^{2-\delta}+\left(\psi^x_{t+\varepsilon'-s}(y)\right)^{2-\delta}\right)\psi^0_s(y)(s\vee 1)dyds \\
&\leq C(t,\delta)t^{1/2}(\varepsilon-\varepsilon')^\delta\int_0^t t(t+\varepsilon')^{-1/2}(t+\varepsilon'-s)^{-1/2-\delta}ds \leq C(t,\delta)t^{3/2}(\varepsilon-\varepsilon')^\delta\left(t+\varepsilon'\right)^{-\delta}.
\end{aligned}$$

□

Thus, we have that

$$\lim_{\varepsilon,\varepsilon'\to 0}\sup_{x\in\mathbb{R},t\geq\eta} E\left[\left(X_t(\psi^x_\varepsilon)-X_t(\psi^x_{\varepsilon'})\right)^2\right]=0, \quad \text{for any } \eta>0. \tag{3.25}$$

By a theorem of Skorohod, we may assume that $X^{(N_k)}$ and X are defined on a common probability space and $X^{N_k} \to X$ in $D([0,\infty), \mathcal{M}_F(\mathbb{R}))$ a.s. Then, from the above arguments, we have that

$$X_t(\psi^x_\varepsilon) = X_0(\psi^x_{t+\varepsilon}) + \tilde{M}_t(\psi^x_{t+\varepsilon-\cdot}) \tag{3.26}$$

for a certain continuous L^2-bounded martingale $\tilde{M}_t(\psi^x_{t+\varepsilon-\cdot})$, where the martingale property of $\tilde{M}_t\left(\psi^x_{t+\varepsilon-\cdot}\right)$ is obtained by the same argument as the proof of Lemma 3.1. Also, we take L^2-limit in (3.26) as $\varepsilon \to 0$ and choose $\varepsilon_n \to 0$ so that for any t and $x \in \mathbb{R}$,

$$\lim_{n\to\infty} X_t(\psi^x_{\varepsilon_n}) = X_0(\psi^x_t) + \tilde{M}_t(\psi^x_{t-\cdot}) \quad \text{a.s. and in } L^2. \tag{3.27}$$

We define $u(t,x) = \lim X_t(\psi^x_{\varepsilon_n})$ for all $t > 0$, $x \in \mathbb{R}$. Standard differential theory shows that for each $t > 0$ with probability 1,

$$X_t(dx) = u(t,x)dx + X^s_t(dx),$$

where X^s_t is a random measure such that $X^s_t(dx) \perp dx$. Also, (3.27) implies that

$$E\left[\int_{\mathbb{R}} u(t,x)dx\right] = \int_{\mathbb{R}} X_0(\psi^x_t)dx = 1 = E\left[X_t(1)\right].$$

Thus, $E\left[X^s_t(1)\right] = 0$ and

$$X_t(dx) = u(t,x)dx, \quad \text{a.s. for all } t > 0.$$

Therefore, we complete the proof of Lemma 3.11 and also Theorem 2.1.

4 Proof of some facts

This section is devoted to the proof of some lemmas used in section 3.

Lemma 4.1. *For any $\beta > 0$ and $K > 0$, we have that*

$$\sup_N E_{Y^1Y^2}\left[\left(1 + \frac{\beta^2}{N^{1/2}}\right)^{\sharp\left\{1\le i\le \lfloor KN\rfloor : Y^1_i = Y^2_i\right\}}\right] < \infty.$$

Also,

$$E_{Y^1Y^2}\left[\left(1 + \frac{\beta^2}{N^{1/2}}\right)^{\sharp\left\{1\le i\le \lfloor KN\rfloor : Y^1_i = Y^2_i\right\}} : Y^1_{\lfloor KN\rfloor} = x, Y^2_{\lfloor KN\rfloor} = y\right]$$
$$\le CK^{1/2}N^{-1/2}\left(P_{Y^1}\left(Y^1_{\lfloor KN\rfloor} = x\right) \wedge P_{Y^1}\left(Y^1_{\lfloor KN\rfloor} = y\right)\right).$$

Proof. First, we remark that

$$E_{Y^1Y^2}\left[\left(1+\frac{\beta^2}{N^{1/2}}\right)^{\sharp\left\{1\le i\le\lfloor KN\rfloor: Y_i^1=Y_i^2\right\}}\right]=E_{Y^1Y^2}\left[\prod_{k=1}^{\lfloor KN\rfloor}\left(1+\frac{\beta^2}{N^{1/2}}\mathbf{1}\left\{Y_k^1=Y_k^2\right\}\right)\right]$$

$$=\sum_{k=0}^{\infty}\frac{\beta^{2k}}{N^{k/2}}\sum_{\mathbf{i}\in D^k(\lfloor KN\rfloor)}\sum_{\mathbf{x}\in\mathbb{Z}^k}P_{Y^1Y^2}\left(Y_{i_j}^1=Y_{i_j}^2=x_j,\ \text{for } 1\le j\le k\right)$$

$$=\sum_{k=0}^{\infty}\frac{\beta^{2k}}{N^{k/2}}\sum_{\mathbf{i}\in D^k(\lfloor KN\rfloor)}\sum_{\mathbf{x}\in\mathbb{Z}^k}P_{Y}\left(Y_{i_j}=x_j,\ \text{for } 1\le j\le k\right)^2 \tag{4.1}$$

where $D^k(\lfloor KN\rfloor)$ is the set defined as

$$D^k(n)=\left\{\mathbf{i}=(i_j)_{j=1}^k\in\mathbb{N}^k:1\le i_1<\cdots<i_k\le n\right\},$$

and the summation for $k>\lfloor KN\rfloor$ is equal to 0.

$$\sum_{\mathbf{i}\in D^k(\lfloor KN\rfloor)}\sum_{\mathbf{x}\in\mathbb{Z}^k}P_{Y}\left(Y_{i_j}=x_j,\ \text{for } 1\le j\le k\right)^2\le C^k\sum_{\mathbf{i}\in D^k(\lfloor KN\rfloor)}\sum_{\mathbf{x}\in\mathbb{Z}^k}\prod_{j=1}^k\frac{P_Y\left(Y_{i_j-i_{j-1}}=x_j-x_{j-1}\right)}{\sqrt{i_j-i_{j-1}}}$$

$$\le C^k\sum_{\mathbf{i}\in D^k(\lfloor KN\rfloor)}\prod_{j=1}^k\frac{1}{\sqrt{i_j-i_{j-1}}}$$

Thus, we have that

$$(4.1)\le\sum_{k=0}^{\infty}\frac{\beta^2k}{N^k}\sum_{\mathbf{i}\in D^k(\lfloor KN\rfloor)}\prod_{j=1}^k\frac{1}{\sqrt{\frac{i_j}{N}-\frac{i_{j-1}}{N}}}. \tag{4.2}$$

Especially,

$$\frac{1}{N^k}\sum_{\mathbf{i}\in D^k(\lfloor KN\rfloor)}\prod_{j=1}^k\frac{1}{\sqrt{\frac{i_j}{N}-\frac{i_{j-1}}{N}}}=\frac{1}{N^k}\sum_{\mathbf{i}\in D^k(\lfloor KN\rfloor)}\prod_{j=1}^k\int_{i_{j-1}/N}^{i_j/N}\frac{dt_{j-1}}{2\sqrt{\frac{i_j}{N}-t_{j-1}}}. \tag{4.3}$$

We will look at the product of integrals in the right hand side. Taking $t_j=t_j'=i_j/N$, It is clear that

$$\frac{1}{N^k}\prod_{j=1}^k\int_{i_{j-1}/N}^{i_j/N}\frac{dt_{j-1}}{2\sqrt{\frac{i_j}{N}-t_{j-1}}}\le\prod_{j=1}^k\int_{t_{j-1}}^{t_j'}\frac{dt_{j-1}}{2\sqrt{t_j'-t_{j-1}}}$$

$$\le\int_{t_{k-1}}^{t_k}\cdots\int_0^{t_1}\frac{1}{2^k}\prod_{j=1}^k\frac{1}{\sqrt{t_j-t_{j-1}}}d\mathbf{t},$$

where the last inequality comes from the followings: Let $F_k(t_1,\cdots,t_k)=\int_{t_{k-1}}^{t_k}\cdots\int_0^{t_1}\prod_{j=1}^k\frac{1}{\sqrt{t_j-t_{j-1}}}d\mathbf{t}$. Then

$$\prod_{j=1}^{k}\int_{t_{j-1}}^{t'_j}\frac{dt_{j-1}}{2\sqrt{t'_j-t_{j-1}}}\le F_k(t_1,\cdots,t_k),\qquad \frac{\partial F_k(t_1,\cdots,t_k)}{\partial t_k}\ge 0\quad\text{for } 0<t_1<\cdots<t_k. \tag{4.4}$$

Indeed, for $k=1$, the equality holds. If for $k=\ell$ (4.4) holds, then

$$\begin{aligned}\prod_{j=1}^{\ell+1}\int_{t_{j-1}}^{t'_j}\frac{dt_{j-1}}{2\sqrt{t'_j-t_{j-1}}}&\le\int_{t_\ell}^{t'_{\ell+1}}\frac{dt_\ell}{2\sqrt{t'_{\ell+1}-t_\ell}}F_\ell(t_1,\cdots,t'_\ell)\\&\le\Big[-\sqrt{t'_{\ell+1}-t_\ell}F_\ell(t_1,\cdots,t_\ell)\Big]_{t_\ell}^{t'_{\ell+1}}+\int_{t_\ell}^{t'_{\ell+1}}\sqrt{t'_{\ell+1}-t_\ell}\frac{\partial F_\ell(t_1,\cdots,t_\ell)}{\partial t_\ell}.\\&=\int_{t_\ell}^{t'_{\ell+1}}\frac{dt_\ell}{2\sqrt{t'_{\ell+1}-t_\ell}}F_\ell(t_1,\cdots,t_\ell)=F_{\ell+1}(t_1,\cdots,t'_{\ell+1}),\end{aligned} \tag{4.5}$$

and the latter part of (4.4) is obtained easily from the second line in (4.5). Thus, we have that

$$\begin{aligned}(4.3)&\le\int_{0<t_1<\cdots<t_k<K}\frac{1}{2^k}\prod_{j=1}^k\frac{1}{\sqrt{t_j-t_{j-1}}}d\mathbf{t}\\&\le\frac{(\pi K)^{k/2}}{2^k\Gamma(\frac{k}{2}+1)},\end{aligned}$$

and since the right hand side is extremely fast decay in k, the right hand side of (4.2) is finite for any β.

Also, the similar argument does hold so that

$$\begin{aligned}&E_{Y^1Y^2}\left[\left(1+\frac{\beta^2}{N^{1/2}}\right)^{\sharp\left\{1\le i\le\lfloor KN\rfloor:Y_i^1=Y_i^2\right\}}:Y^1_{\lfloor KN\rfloor}=x,Y^2_{\lfloor KN\rfloor}=y\right]\\&=\sum_{k=1}^\infty\frac{\beta^{2(k-1)}}{N^{(k-1)/2}}\sum_{\mathbf{i}\in D^{k-1}(\lfloor KN\rfloor-1)}\sum_{\mathbf{x}\in\mathbb{Z}^{k-1}}\begin{pmatrix}P_Y\left(Y_{i_j}=x_j,\text{ for }1\le j\le k-1,Y_{\lfloor KN\rfloor}=x\right)\\\times P_Y\left(Y_{i_j}=x_j,\text{ for }1\le j\le k-1,Y_{\lfloor KN\rfloor}=y\right)\end{pmatrix}\\&\quad+\sum_{k=1}^\infty\frac{\beta^{2k}}{N^{k/2}}\sum_{\mathbf{i}\in D^{k-1}(\lfloor KN\rfloor-1)}\sum_{\mathbf{x}\in\mathbb{Z}^{k-1}}\begin{pmatrix}P_Y\left(Y_{i_j}=x_j,\text{ for }1\le j\le k-1,Y_{\lfloor KN\rfloor}=x\right)\\\times P_Y\left(Y_{i_j}=x_j,\text{ for }1\le j\le k-1,Y_{\lfloor KN\rfloor}=y\right)\end{pmatrix}\\&\le\sum_{k=1}^\infty 2C^k\frac{\beta^{2(k-1)}}{N^{(k-1)/2}}\sum_{\mathbf{i}\in D^{k-1}(\lfloor KN\rfloor-1)}\left(\prod_{j=1}^{k-1}\frac{1}{\sqrt{i_j-i_{j-1}}}\right)\frac{P_Y\left(Y^1_{\lfloor KN\rfloor}=x\right)\wedge P_Y\left(Y^1_{\lfloor KN\rfloor}=y\right)}{\sqrt{\lfloor KN\rfloor-i_{k-1}}}\end{aligned}$$

$$\leq \sum_{k=1}^{\infty} \frac{C^k\beta^{2(k-1)}}{N^{1/2}} \frac{P_Y\left(Y^1_{\lfloor KN\rfloor}=x\right)\wedge P_Y\left(Y^1_{\lfloor KN\rfloor}=y\right)}{N^{k-1}} \sum_{\mathbf{i}\in D^{k-1}(\lfloor KN\rfloor)} \prod_{j-1}^{k-1} \frac{1}{\sqrt{\frac{i_j}{N}-\frac{i_{j-1}}{N}}} \frac{1}{\sqrt{K-\frac{i_{k-1}}{N}}}$$

$$\leq \frac{P_Y\left(Y^1_{\lfloor KN\rfloor=x}\right)\wedge P_Y\left(Y^1_{\lfloor KN\rfloor}=y\right)}{N^{1/2}} \sum_{k=1}^{\infty} C^{2k}\beta^{2(k-1)} \frac{(\pi K)^{k/2}}{2^{\frac{k+1}{2}}\Gamma\left(\frac{k}{2}+1\right)}$$

$$\leq C\frac{P_Y\left(Y^1_{\lfloor KN\rfloor=x}\right)\wedge P_Y\left(Y^1_{\lfloor KN\rfloor}=y\right)}{N^{1/2}}K^{1/2}$$

□

Since the next lemma is a general theory for branching random walks in random environment, we introduce new notations. Let $\omega = \left\{\{q_{n,x}(k)\}_{k=0}^{\infty} : (n,x)\in\mathbb{N}\times\mathbb{Z}^d\right\}$ be the offspring distributions assigned to each site (n,x) which are i.i.d. in (n,x). Branching random walks in random environment under fixed environment ω is defined by the following rule:

i) There exists just "one" particle at the origin at time 0.

ii) Each particle located at site x at time n independently of each others chooses a nearest neighbor site and uniformly and moves there at time $n+1$. Then, it is replaced by k-children with probability $q_{n,x}(k)$ independently of each others

We denote by B_n and by $B_{n.x}$ the total number of particles at time n and the local number of particles located at site x at time n. Also, we denote by $m_{n,x}^{(p)}$ the p-th moment of offsprings for offspring distribution $\{q_{n,x}(k)\}$, that is

$$m_{n,x}^{(p)} = \sum_{k=0}^{\infty} k^p q_{n,x}(k).$$

This model is called branching random walks in random environment which is studied well, in particular, some properties as measure valued processes for "supercritical" case[3, 10, 9]. Also, the continuous counterpart, branching Brownian motion in random environment is introduced by Shiozawa[17, 18]. In particular, we know that the normalized random measure weakly converges to Gaussian measure in probability in one phase, whereas the localization has occurred in the other phase.

Lemma 4.2. *If* $E[m_{n,x}^{(p)}]=K<\infty$ *for* $p\in\mathbb{N}$ *and* $E\left[m_{n,x}^{(1)}\right]=1$, *then*

$$E\left[B_n^p\right] \leq C(p,K)n^{p-1}E_{Y^1\cdots Y^p}\left[E\left[\left(m_{0,0}^{(1)}\right)^p\right]^{\sharp\left\{1\leq i\leq n: Y_i^a=Y_i^b, a\neq b\in\{1,\cdots,p\}\right\}}\right]$$

and

$$E\left[\prod_{i=1}^p B_{n,x_i}\right] \leq C(p,K)n^{p-1}E_{Y^1\cdots Y^p}\left[E\left[\left(m_{0,0}^{(1)}\right)^p\right]^{\sharp\left\{1\leq i\leq n: Y_i^a=Y_i^b, a\neq b\in\{1,\cdots,p\}\right\}} : Y_n^i = x_i \text{ for } 1\leq i\leq p\right].$$

Before giving a proof, we give another representation of B_n. Let $\{V^{\mathbb{x}}_{n,x} : \mathbb{x} \in \mathcal{T}(n,x) \in \mathbb{N} \times \mathbb{Z}^d\}$ be $\mathbb{N}$-valued random variables with $P\left(V^{\mathbb{x}}_{n,x} = k \middle|\, \omega\right) = q_{n,x}(k)$. Let $\{X^{\mathbb{x}}_{n,x} : \mathbb{x} \in \mathcal{T}, (n,x) \in \mathbb{N} \times \mathbb{Z}^d\}$ be i.i.d. random variables with $P(X^{\mathbb{x}}_{n,x} = e) = \frac{1}{2d}$ for $e = \pm e_j$, $j = 1, \cdots, d$ where e_j are unit vector on $\mathbb{Z}^d$.

We consider the event {particle $\mathbb{y}$ exists and locates at site y at time $|\mathbb{x}| = n$} and its indicator function

$$B^{\mathbb{y}}_{n,y} = \mathbf{1}\,\{\text{particle } \mathbb{y} \text{ exists and locates at site } y \text{ at time } |\mathbb{x}| = n\}\,.$$

Then, it is clear that

$$\begin{aligned}
B^{\mathbb{x}}_{0,x} &= \delta_{x,\mathbb{x}} = \begin{cases} 1 & \text{if } x = 0 \text{ and } \mathbb{x} = \mathbf{1} \\ 0 & \text{otherwise} \end{cases} \\
B^{\mathbb{y}}_{n,y} &= \sum_{x,\mathbb{x}} B^{\mathbb{x}}_{n-1,x} \mathbf{1}\left\{X^{\mathbb{x}}_{n-1,x} = y - x, V^{\mathbb{y}}_{n-1,x} \geq \mathbb{y}/\mathbb{x} \geq 1\right\} \\
&= \sum_{0 \to y} \sum_{\mathbf{1} \to \mathbb{y}} \prod_{i=0}^{n-1} \mathbf{1}\left\{X^{\mathbb{y}_i}_{i,y_i} = y_{i+1} - y_i, V^{\mathbb{y}_i}_{i,y_i} \geq \mathbb{y}_{i+1}/\mathbb{y}_i \geq 1\right\},
\end{aligned}$$

and

$$B_{n,y} = \sum_{\mathbb{y}} \sum_{0 \to y} \sum_{\mathbf{1} \to \mathbb{y}} \prod_{i=0}^{n-1} \mathbf{1}\left\{X^{\mathbb{y}_i}_{i,y_i} = y_{i+1} - y_i, V^{\mathbb{y}_i}_{i,y_i} \geq \mathbb{y}_{i+1}/\mathbb{y}_i \geq 1\right\}.$$

We introduce new Markov chain $\mathbf{Y} = (Y, \mathbb{Y})$ on $\mathbb{Z}^d \times \mathcal{T}$ which are determined by

$Y_0 = 0, \mathbb{Y}_0 = \mathbf{1} \in T_0$.

$$P_{Y\mathbb{Y}}\left(Y_{n+1} = y, \mathbb{Y}_{n+1} = \mathbb{y} \middle|\, Y_n = x, \mathbb{Y}_n = \mathbb{x}\right) = \begin{cases} \frac{1}{2d} \sum_{k \geq \mathbb{y}/\mathbb{x}} q(k) & \text{if } |y - x| = 1, \mathbb{y}/\mathbb{x} < \infty \\ 0 & \text{otherwise,} \end{cases}$$

where $q(k) = E[q_{n,x}(k)]$. Let $A^{\mathbb{x},\mathbb{y}}_{n,x,y} = \mathbf{1}\left\{X^{\mathbb{x}}_{n,x} = y - x, V^{\mathbb{x}} \geq \mathbb{y}/\mathbb{x}\right\}$. Then, we have the following representation of $B_{n,y}$ [15]:

$$B_{n,y} = E_{Y\mathbb{Y}}\left[\prod_{i=0}^{n-1} \frac{A^{\mathbb{Y}_i,\mathbb{Y}_{i+1}}_{i,Y_i,Y_{i+1}}}{E\left[A^{\mathbb{Y}_i,\mathbb{Y}_{i+1}}_{i,Y_i,Y_{i+1}}\right]} : Y_n = y\right]$$

and

$$E\left[\prod_{i=1}^{p} B_{n,x_i}\right] = E_{\mathbf{Y}^1 \cdots \mathbf{Y}^p}\left[\prod_{i=0}^{n-1} E\left[\frac{\prod_{j=1}^{p} A^{\mathbb{Y}^j_i,\mathbb{Y}^j_{i+1}}_{i,Y^j_i,Y^j_{i+1}}}{\prod_{j=1}^{p} E\left[A^{\mathbb{Y}^j_i,\mathbb{Y}^{j+1}_{i+1}}_{i,Y^j_i,Y^j_{i+1}}\right]}\right] : Y^i_n = x_i \text{ for } 1 \leq i \leq p\right]$$

$$E\left[B_n^p\right]=E_{\mathbf{Y}^1\cdots\mathbf{Y}^p}\left[\prod_{i=0}^{n-1}E\left[\frac{\prod_{j=1}^p A_{i,Y_i^j,Y_{i+1}^j}^{\mathbb{Y}_i^j,\mathbb{Y}_{i+1}^j}}{\prod_{j=1}^p E\left[A_{i,Y_i^j,Y_{i+1}^j}^{\mathbb{Y}_i^j,\mathbb{Y}_{i+1}^{j+1}}\right]}\right]\right]. \tag{4.6}$$

Proof of Lemma 4.2. We remark the following facts:

i) If $y\neq y'$, then $A_{i,x,y}^{\mathbb{x},\mathbb{y}}A_{i,x,y'}^{\mathbb{x},\mathbb{y}'}=0$ almost surely. Especially, for $\left\{\mathbf{Y}_i^j: i=0,\cdots,n\right\}$ and $\left\{\mathbf{Y}_i^{j'}: i=0,\cdots,n\right\}$, if there exists an i such that $\mathbf{Y}_i^j=\mathbf{Y}_i^{j'}$ and $Y_{i+1}^j\neq Y_{i+1}^{j'}$, then

$$\prod_{i=0}^{n-1}E\left[\frac{\prod_{j=1}^p A_{i,Y_i^j,Y_{i+1}^j}^{\mathbb{Y}_i^j,\mathbb{Y}_{i+1}^j}}{\prod_{j=1}^p E\left[A_{i,Y_i^j,Y_{i+1}^j}^{\mathbb{Y}_i^j,\mathbb{Y}_{i+1}^{j+1}}\right]}\right]=0,$$

almost surely.

ii) If $\mathbb{y}/\mathbb{x}=k$, $\mathbb{y}'/\mathbb{x}=\ell$, and $k\leq\ell$, then $A_{i,x,y}^{\mathbb{x},\mathbb{y}}A_{i,x,y}^{\mathbb{x},\mathbb{y}'}=A_{i,x,y}^{\mathbb{x},\mathbb{y}}$ almost surely.

iii) If $\{\mathbb{x}^j: j=1,\cdots,p\}$ are different from each other and $\mathbb{y}^j/\mathbb{x}^j=k_j$, then $E\left[\prod_{j=1}^r A_{i,x^j,y^j}^{\mathbb{x}^j,\mathbb{y}^j}\right]=\left(\frac{1}{2d}\right)^p\sum_{s_1\geq k_1}\cdots\sum_{s_p\geq k_p}E\left[\prod_{j=1}^p q_{i,x}(s_j)\right]$.

Thus, the possible cases are the followings:

$$E\left[E_{\mathbf{Y}^1\cdots\mathbf{Y}^p}\left[\frac{\prod_{j=1}^p A_{i,Y_i^j,Y_{i+1}^j}^{\mathbb{Y}_i^j,\mathbb{Y}_{i+1}^j}}{\prod_{j=1}^p E\left[A_{i,Y_i^j,Y_{i+1}^j}^{\mathbb{Y}_i^j,\mathbb{Y}_{i+1}^j}\right]}\middle| Y_i^j=x^j,\mathbb{Y}_i^j=\mathbb{x}^j \text{ for } j=1,\cdots p\right]\right]$$
$$=\begin{cases}1 & x^j \text{ are different from each others.}\\ E\left[\prod_{j=1}^r m_{i,y^j}^{(1)}\right] & \text{if } \mathbb{x}^j \text{ are different from each others}\\ (A), & \end{cases}$$

where (A) is the other case which we will look at in more detail. We divide the set $\{1,\cdots,p\}$ into the disjoint union such that

$$\{1,\cdots,p\}=\coprod_{k=j_1}^{j_p}I_k, \tag{4.7}$$

where $I_k=\{j\in\{1,\cdots,p\}:\mathbb{x}^j=\mathbb{x}^k\}$ and $j_1,\cdots j_p$ is the set of index of equivalence class I_k. For $\mathbb{y}^j/\mathbb{x}^j=k_j$, we set $K_{j_\ell}=\min\{j\in I_{j_\ell}\}$. Then, we have that

$$E\left[E_{\mathbf{Y}^1\cdots\mathbf{Y}^p}\left[\frac{\prod_{j=1}^p A_{i,Y_i^j,Y_{i+1}^j}^{\mathbb{Y}_i^j,\mathbb{Y}_{i+1}^j}}{\prod_{j=1}^p E\left[A_{i,Y_i^j,Y_{i+1}^j}^{\mathbb{Y}_i^j,\mathbb{Y}_{i+1}^j}\right]}\mathbf{1}\left\{\mathbb{Y}_{i+1}^j=\mathbb{y}^j \text{ for } j=1,\cdots,p\right\}\middle| Y_i^j=x^j,\mathbb{Y}_i^j=\mathbb{x}^j \text{ for } j=1,\cdots,p\right]\right]$$

$$= E\left[\prod_{\ell=j_1}^{j_p}\left(\sum_{k\geq k_\ell} q_{i,x^\ell}(k)\right)\right].$$

By the above argument, we find that $\mathbf{Y}^1, \cdots, \mathbf{Y}^P$ evolves according the following steps:

i) First, the set process $\{S(m) : m = 0, \cdots, n\}$ starts from the set $I^{(0)} = \{1, \cdots, p\}$ until time $i^{(1)}$, and then it splits into some sets $I^{(1,1)}, \cdots, I^{(1,k^{(1)})}$. ($i^{(1)}$ is the last time when $\mathbf{Y}_i^j$ coincide and $I^{(1,1)}, \cdots, I^{(1,k^{(1)})}$ are the equivalent class defined in (4.7) for $\mathbb{Y}^j_{i^{(0)}+1}$).

ii) When the set process $S(m) = \{I^{(\ell,1)}, \cdots, I^{(\ell,k^{(\ell)})}\}$, it jumps to the new sets $\{I^{(\ell+1,1)}, \cdots, I^{(\ell+1,k^{(\ell+1)})}\}$ where each $I^{(\ell+1,r)}$ is a partition of some set of $I^{(\ell,1)}, \cdots I^{(\ell,k^{(\ell)})}$ at some time $i^{(\ell)}$. ($\mathbf{Y}^{(j)}$ $j \in I^{(\ell,s)}$ for each $s = 1, \cdots, k^{(\ell)}$ coincides until time $i^{(\ell)}$ and $Y^j_{i^{(\ell)}+1} \neq Y^{j'}_{i^{(\ell)}+1}$ for some $j, j' \in I^{(\ell,k)}$ for some k).

iii) If $S(m) = \{\{1\}, \cdots, \{p\}\}$, then $S(m) = S(m')$ for $m' \geq m$.

First, we remark that the combination of $i^{(1)}, \cdots, i^{(p-1)}$ (it may stops for less steps) are at most n^p-th order. Also,

$$E\left[E_{\mathbf{Y}^1\cdots\mathbf{Y}^p,S}\left[\frac{\prod_{j=1}^p A^{\mathbb{Y}_i^j,\mathbb{Y}_{i+1}^j}_{i,Y_i^j,Y_{i+1}^j}}{\prod_{j=1}^p E\left[A^{\mathbb{Y}_i^j,\mathbb{Y}_{i+1}^j}_{i,Y_i^j,Y_{i+1}^j}\right]}\mathbf{1}\left\{i^{(\ell)} = i\right\}\middle| Y_i^j = x^j, \mathbb{Y}_i^j = \mathbb{x}^j \text{ for } j = 1, \cdots, p\right]\right]$$
$$\leq C(p)K,$$

and

$$E\left[E_{\mathbf{Y}^1\cdots\mathbf{Y}^p,S}\left[\frac{\prod_{j=1}^p A^{\mathbb{Y}_i^j,\mathbb{Y}_{i+1}^j}_{i,Y_i^j,Y_{i+1}^j}}{\prod_{j=1}^p E\left[A^{\mathbb{Y}_i^j,\mathbb{Y}_{i+1}^j}_{i,Y_i^j,Y_{i+1}^j}\right]}\mathbf{1}\left\{i^{(\ell)} \neq i, \text{ for } \ell = 1, \cdots, p\right\}\middle| Y_i^j = x^j, \mathbb{Y}_i^j = \mathbb{x}^j \text{ for } j = 1, \cdots, p\right]\right]$$
$$\leq \prod_{k\in\mathcal{K}} E\left[(m_{i,x^k})^{\sharp\left\{j:x^j=x^k\right\}}\right] \leq \prod_{k\in\mathcal{K}} E\left[(m_{i,x^k})^p\right]^{\sharp\{j:x_j=x_k\}/p} \leq E\left[(m_{i,x^k})^p\right]^{\mathbf{1}\left\{x^j=x^k, \text{ for some } j\neq k\right\}},$$

where $\mathcal{K}$ be the set of index for equivalence class $\left\{j : x^j = x^k\right\}$.

Thus, we have that

$$E\left[B_n^P\right] \leq C(p,K)n^{(p-1)}E_{\mathbf{Y}^1\cdots\mathbf{Y}^p}\left[E\left[(m_{n,x})^p\right]^{\sharp\left\{i\leq n: Y_i^j = Y_i^{j'} \text{ for } j\neq j' \in\{1,\cdots,p\}\right\}}\right].$$

The latter part of Lemma 4.2 can be proved by the same argument.

□

Corollary 4.3. *Under the same assumption in Lemma 4.2,*

$$
\begin{aligned}
&E\left[\prod_{j=1}^{q}\prod_{i=1}^{p_j} B^{(j)}_{n,x_{(j,i)}}\right] \\
&\leq C(\mathbf{p},K) n^{\left(\sum_{j=1}^q p_j - q\right)} E_{(\mathbf{Y}^{j,i})}\left[E\left[(m_{0,0})^{\sum_{j=1}^q p_j}\right]^{\sharp\left\{\substack{k\leq n: Y_k^{j_1,i_1}=Y_k^{j_2,i_2},\ for \\ (j_1,i_1)\neq(j_2,i_2)\in\{(j,i):j=1,\cdots,q,i=1,\cdots,p_j\}}\right\}} : Y_n^{(j,i)} = x_{j,i}\right],
\end{aligned}
$$

where $B^{(j)}_{n,x}$ is the number of particles from initial particle j at site x at time n.

Proof. If we regard $i^{(1)} = -1$ and $S(0) = \{\{1,\cdots,p_1\},\cdots,\{\sum_{j=1}^{q-1} p_j+1,\cdots,\sum_{j=1}^q p_j\}\}$, then $S(m)$ stops at $\{\{1\},\cdots,\{\sum_{j=1}^q p_j\}\}$ at most $\sum_{j=1}^q p_j - q$ jumps. $\square$

Also, by the same argument in the proof of Theorem 2.1, we can obtain more general super-Brownian motion in random environment under the following setting.

Let $\{q^{N)}_{n,x} : (n,x) \in \mathbb{N}\times\mathbb{Z}\}_{N\in\mathbb{N}}$ be the sequences of i.i.d. offspring distributions. Let $X_0^{(N)}$ be the sequences of the measures on $\mathbb{R}$ which is represented as

$$X_0^{(N)} = \sum_{i=0}^{M_N} \delta_{x_i/N^{1/2}},$$

where $x_i \in 2\mathbb{Z}$ is the set of site where i-th particle locates.

Corollary 4.4. *Suppose $X_0^{(N)} \Rightarrow \nu$ for $\nu \in \mathcal{M}_F(\mathbb{R})$. If we assume that*

$$\sup_N E\left[\sum_{k=0}^\infty k^4 q^{(N)}_{n,x}(k)\right] < \infty,$$
$$E[\sum_{k=0}^\infty k^2 q^{(N)}_{n,x}(k)] \to \gamma+1,$$

and there exists i.i.d. random variables $\{\xi_{n,x} : (n,x)\in\mathbb{N}\times\mathbb{Z}\}$ with $E[\xi(n,x)] = 0$ and $E[\xi(n,x)^2] = 1$ such that

$$\sum_{k=0}^\infty k\, q_{n,x}(k) = 1 + \frac{\beta\xi(n,x)}{N^{1/4}} \quad \textit{for } \beta\in\mathbb{R}.$$

Then, the sequences of the measure-valued processes $\{X^{(N)}_\cdot : N \in \mathbb{N}\}$ is C-relatively compact and its limit point is a solution of the following martingale

problem:

$$\begin{cases} \text{For all } \phi \in \mathcal{D}(\Delta), \\ \qquad\qquad Z_t(\phi) = X_t(\phi) - \nu(\phi) - \frac{1}{2}\int_0^t X_s(\Delta\phi)ds \\ \text{is an } \mathcal{F}_t^X\text{-continuous square-integrable martingale such that } Z_0(\phi) = 0 \text{ and} \\ \qquad\qquad \langle Z(\phi)\rangle_t = \int_0^t X_s(\gamma\phi^2)ds \\ \qquad\qquad\qquad + \frac{\beta^2}{2}\int_0^t \int_{\mathbb{R}\times\mathbb{R}} \delta_{x-y}\phi(x)\phi(y)X_s(dx)X_s(dy)ds. \end{cases}$$

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